

Lecture 4

5. Derivations and tangent vectors

In this lecture, we see another way of defining tangent vectors and also talk about directional derivatives which is the analogue of the notion of partial derivatives from multi-variable calculus. This will be useful in doing actual computations.

Notation. We will denote by $C^k(M) = \{f : M \rightarrow \mathbb{R} \mid f \text{ is } C^k\}$. We also denote by $C_p^\infty = \bigcup_{U \text{ open, } p \in U} C^\infty(U)$.

Definition 5.1 (Directional derivatives). Let M be a smooth manifold and let $p \in M$. Let $\gamma'(0) \in T_p M$ and $f \in C^\infty(M)$. We define

$$\partial_{\gamma'(0)} f = df_p(\gamma'(0)) = \frac{d}{dt}(f \circ \gamma)|_{t=0} \in \mathbb{R},$$

and call it the **directional derivative of f** in the direction of the tangent vector $\gamma'(0)$.

Definition 5.2 (Derivations). A map $\partial : C_p^\infty \rightarrow \mathbb{R}$ is called a derivation at p if the following holds.

1. Locality: If $\tilde{U} \subset U$ is open, $p \in \tilde{U}$ and $f \in C^\infty(U)$ then

$$\partial f = \partial(f|_{\tilde{U}}).$$

2. \mathbb{R} -linearity: if $a, b \in \mathbb{R}$ and $f, g \in C_p^\infty$ then

$$\partial(af + bg) = a\partial f + b\partial g.$$

3. Leibniz rule: If $f, g \in C_p^\infty$ then

$$\partial(fg) = g\partial f + f\partial g.$$

Clearly, the usual partial derivatives $\frac{\partial}{\partial x^i}$, $i = 1, \dots, n$ in \mathbb{R}^n are derivations. In fact, the definition was made to prove the following.

Proposition 5.3. *The directional derivatives $\partial_{\gamma'(0)}$ on a smooth manifold are derivations.*

Proof. We just check the Leibniz rule which essentially follows from the Leibniz rule in \mathbb{R}^n . We have

$$\begin{aligned} \partial_{\gamma'(0)}(fg) &= \frac{d}{dt}((fg) \circ \gamma)|_{t=0} \\ &= \frac{d}{dt}((f \circ \gamma)(g \circ \gamma))|_{t=0} \\ &= (g \circ \gamma)(0) \frac{d}{dt}(f \circ \gamma)|_{t=0} + (f \circ \gamma)(0) \frac{d}{dt}(g \circ \gamma)|_{t=0} \quad (\text{Leibniz rule in Euclidean spaces}) \\ &= g(p)\partial_{\gamma'(0)}f + f(p)\partial_{\gamma'(0)}g. \end{aligned}$$

□

We denote the set of **all derivations at p** by $\text{Der}(C_p^\infty)$. This space is a real vector space via

$$(\alpha\partial_1 + \beta\partial_2)(f) = \alpha\partial_1 f + \beta\partial_2 f.$$

We want another way to understand tangent vectors and the next lemma exactly starts with that fact.

Lemma 5.4. *Let M be a smooth manifold and let $p \in M$. The map*

$$\partial : T_p M \rightarrow \text{Der}(C_p^\infty), \quad \gamma'(0) \mapsto \partial_{\gamma'(0)},$$

is linear.

Proof. Let $\phi : U \rightarrow V$ be a chart of M with $p \in U$. We will have to use the vector structure on $T_p M$ to check the linearity of the map, which is clearly a well-defined map. That is, we have to show that

$$\partial \circ (d\phi|_p)^{-1}$$

is linear. Assume $v \in \mathbb{R}^n$ and put

$$\gamma(t) := \phi^{-1}(\phi(p) + tv).$$

For $f \in C_p^\infty$, we compute,

$$\begin{aligned} (\partial \circ (d\phi|_p)^{-1}(v))(f) &= df|_p((d\phi|_p)^{-1}(v)) \\ &= df|_p(\gamma'(0)) \\ &= \left. \frac{d}{dt}(f \circ \gamma(t)) \right|_{t=0} \\ &= \left. \frac{d}{dt}(f \circ \phi^{-1}(\phi(p) + tv)) \right|_{t=0} \\ &= \langle D(f \circ \phi^{-1})|_{\phi(p)}, v \rangle. \end{aligned}$$

which is an expression that is linear in v . □

Thus, we can view tangent vectors as derivations and the way to think about it is that you take a tangent vector and take the directional derivative in its direction.

Let e_1, \dots, e_n be the standard basis of \mathbb{R}^n . Then $(d\phi|_p)^{-1}(e_1), \dots, (d\phi|_p)^{-1}(e_n)$ form a basis of $T_p M$. What does this basis look like? We compute,

$$\partial_{(d\phi|_p)^{-1}(e_j)}(f) = \langle D(f \circ \phi^{-1})|_{\phi(p)}, e_j \rangle = \left. \frac{\partial(f \circ \phi^{-1})}{\partial x^j} \right|_{\phi(p)} =: \left. \frac{\partial f}{\partial x^j} \right|_p$$

where we are abusing notation and still writing $f \circ \phi^{-1} = f$ which is the coordinate representation of f in the chart (U, ϕ) .

This tells us that if (U, ϕ) is any chart on M and we write (x^1, \dots, x^n) as the coordinates on U then

For every chart (U, ϕ) , we have the derivations

$$\left. \frac{\partial}{\partial x^1} \right|_p, \dots, \left. \frac{\partial}{\partial x^n} \right|_p.$$