4. Tangent vectors and Tangent spaces

Recall that the definition of a smooth manifold as well as a smooth map was motivated from the point of view of defining what a function f being *differentiable* could mean. We again go back to our study of differential calculus and recall that the derivative of a map was the *best linear approximation* of the map. In Euclidean spaces, a linear approximation makes sense both theoretically and pictorially. What could be the appropriate notion of a linear approximation on a manifold? Let's start with the simplest possible case.

4.1 Tangent vector and tangent spaces

We first recall that if $p \in M$ then by a curve on M passing through p, we mean a map $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$. If the map is differentiable/smooth then we say that the curve is differentiable/smooth.

Definition 4.1. A **tangent vector** on M at the point $p \in M$ is an **equivalence class** of differentiable curves $[\gamma]$ passing through p where $\gamma_1 \sim \gamma_2$ if for a chart $\phi : U \to V$ with $p \in U$, we have

$$\frac{d}{dt}(\phi \circ \gamma_1)_{|t=0} = \frac{d}{dt}(\phi \circ \gamma_2)_{|t=0}. \tag{4.1}$$

We denote $[\gamma] = \gamma'(0)$.

It's obvious that $\sim$ is indeed an equivalence relation. We also note that this definition is independent of the choice of chart. If (U', ψ) is another chart containing p then we use the chain rule to compute

$$\frac{d}{dt}(\psi \circ \gamma)_{|t=0} = \frac{d}{dt}\left((\psi \circ \phi^{-1}) \circ (\phi \circ \gamma)\right)_{|t=0} = D(\psi \circ \phi^{-1}) \circ \frac{d}{dt}(\phi \circ \gamma)_{|t=0},$$

where $D(\psi \circ \phi^{-1})(\phi(p)) : \mathbb{R}^n \to \mathbb{R}^n$ is the derivative of the transition map $\psi \circ \phi^{-1}$ at the point $\phi(p)$ and is an invertible linear map since $\psi \circ \phi^{-1}$ is a smooth map with a smooth inverse. Thus, the condition

$$\frac{d}{dt}(\phi \circ \gamma_1)_{|t=0} = \frac{d}{dt}(\phi \circ \gamma_2)_{|t=0}$$

is equivalent to the condition

$$\frac{d}{dt}(\psi \circ \gamma_1)_{|t=0} = \frac{d}{dt}(\psi \circ \gamma_2)_{|t=0}.$$

Definition 4.2. Let M be a smooth manifold and $p \in M$. The set

$$T_pM = \{\gamma'(0) \mid \gamma \text{ is a curve passing through } p\}$$

is called the **tangent space** of M at p.

Even though the definition of a tangent space is very geometric, it doesn't give much idea about the type of space T_pM is. The next proposition shows that it is a n-dimensional vector space.

Proposition 4.3. *Let M^n be a smooth manifold and $p \in M$ with chart (U, ϕ) containing p. The tangent space T_pM has a unique vector space structure such that the map*

$$d\phi_p : T_pM \to \mathbb{R}^n, \ [\gamma] \mapsto (\phi \circ \gamma)'(0) \tag{4.2}$$

is an isomorphism of vector spaces. Thus, for every point $p \in M$, the tangent space T_pM at that point is a n-dimensional real vector space.

Proof. The map in (4.2) is clearly well-defined and injective. For surjectivity, we'd need to show that if $v \in \mathbb{R}^n$ then there exists a curve whose tangent vector is v. Define $\gamma(t) = \phi^{-1}(\phi(p) + tv)$ and choose $\varepsilon > 0$ small enough so that $\phi(p) + tv \in \phi(U)$. Then

$$d\phi_p(\gamma'(0)) = \frac{d}{dt}\left(\phi \circ \phi^{-1}(\phi(p) + tv)\right)_{|t=0} = \frac{d}{dt}(\phi(p) + tv)_{|t=0} = v.$$

The map is also a linear isomorphism and in fact, the vector space structure is explicitly described as follows: let $a, b \in \mathbb{R}$ and $\gamma_1'(0), \gamma_2'(0) \in T_pM$ then

$$a\gamma_1'(0) + b\gamma_2'(0) = (d\phi_p)^{-1}\left(ad\phi_p(\gamma_1'(0)) + bd\phi_p(\gamma_2'(0))\right).$$

So the only remaining thing to prove is that the vector space structure on T_pM is independent of the chart. To see this, let $(\psi : \tilde{U} \to \tilde{V})$ be another chart with $p \in \tilde{U}$. We want to show that the map $d\psi_p : T_pM \to \mathbb{R}^n$ is linear with respect to the vector space structure induced by ϕ. But we have already seen that

$$d\psi_p = \underbrace{D(\psi \circ \phi^{-1})_{\phi(p)}}_{\text{invertible linear}} \circ \underbrace{d\phi_p}_{\text{invertible linear}},$$

and hence is a vector space isomorphism. □

We can think of T_pM as the linear approximation to M at the point p.

We can now define the differential of a differentiable map between smooth manifolds.

Lemma 4.4. *Let M and N be smooth manifolds and let $p \in M$. Let $f : M \to N$ be a smooth map near p. Then the map*

$$df_p : T_pM \to T_{f(p)}N, \ \gamma'(0) \mapsto (f \circ \gamma)'(0), \tag{4.3}$$

is a well-defined linear map between vector spaces T_pM and $T_{f(p)}N$.
The map df_p is called the **differential of f at the point** *p.*

Proof. Let (U, ϕ) be a chart of M containing p and let $(\tilde{U}, \psi)$ be a chart of N containing $f(p)$. We compute, using the chain rule,

$$\begin{aligned} d\psi_{f(p)}((f \circ \gamma)'(0)) &= (\psi \circ f \circ \gamma)'(0) \\ &= \left((\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \gamma)\right)'(0) \\ &= D(\psi \circ f \circ \phi^{-1})_{\phi(p)} \cdot ((\phi \circ \gamma)'(0)) \\ &= D(\psi \circ f \circ \phi^{-1})_{\phi(p)} \cdot d\phi_p(\gamma'(0)). \end{aligned}$$

The previous equation implies that

$$df_p = \left(d\psi_{f(p)}\right)^{-1} \circ D(\psi \circ f \circ \phi^{-1})_{\phi(p)} \cdot d\phi_p,$$

and hence is well-defined, that is, independent of the choice of curve γ and is a linear map. □

We notice that if $U \subset M$ is open then the differential of the inclusion map $i : U \hookrightarrow M$ is the natural isomorphism $di : T_pU \to T_pM$ given by

$$\gamma'(0) \mapsto (i \circ \gamma)'(0) = \gamma'(0),$$

and hence $T_pU = T_pM$.

Exercise 4.5. Prove that if M, N are two smooth manifolds and $p \in M, q \in N$, then

$$T_{(p,q)}(M \times N) \cong T_pM \times T_qN$$

in a canonical way.

Recall that V is a vector space then its dual space is denoted by V^* and is the space of linear functionals on V, i.e., scalar-valued linear maps on V. In other words,

$$V^* = Hom(V, \mathbb{R}), \ \text{ where } \ Hom(V, W) = \{f : V \to W \mid f \text{ linear}\}.$$

Thus, we define the **cotangent space** at $p \in M$ as

$$T_p^*(M) = Hom(T_pM, \mathbb{R}) = (T_pM)^*. \tag{4.4}$$

The previously discussed spaces T_pM and T_p^*M, when taken together for all points of M, themselves have a manifold structure and are useful examples of mathematical objects called **vector bundles**. We will see more about them much later but for now, let's discuss tangent and cotangent bundles.

4.2 Tangent and cotangent bundle

Definition 4.6. The **tangent bundle** TM of a smooth manifold M is the (disjoint) union of all of its tangent spaces:

$$TM = \bigcup_{p \in M} T_pM.$$

• The map $\pi : TM \to M$ which maps a tangent space $T - pM$ to the point p is called the **projection map.** Clearly, π is surjective.
• The subset of TM which consists of the zero vectors $0 \in T_pM,\ p \in M$ is called the **zero-section** of TM.
• The tangent space $T_pM \subset TM$ is called the **fiber** over $p \in M$.

Thus, every fiber of the tangent bundle is an n-dimensional real vector space. Even though, TM looks like an arbitrary set right now, the result below proves that not only it has a topology inherited from M, but it is a smooth manifold in its own right.

Remark 4.7. An arbitray point of TM looks like (p, v) with $p \in M$ and $v \in T_pM$ where the latter is the tangent space at the point p.

Theorem 4.8. *Let M^n be a smooth manifold. The tangent bundle TM can be endowed naturally with a smooth structure from M making it into a $2n$-dimensional smooth manifold. With respect to the smooth structure, the projection map $\pi : TM \to M$, the inclusion map $i : M \hookrightarrow TM$ as the zero-section and the natural inclusions $i : T_pM \hookrightarrow TM, p \in M$ are all smooth maps.*

Proof. The first step in describing a smooth structure on TM is to provide an atlas.

Claim 4.9. *Let (U, ϕ) be a chart on M. Then this gives a chart $(TU, T\phi)$ on TM where $TU = \underset{p \in U}{T_pM}$ and $T\phi : TU \to \mathbb{R}^{2m}$ is defined using the linear isomorphisms $d\phi_p : T_pM \to \mathbb{R}^n$ by*

$$TU \supset T_pM \ni X \mapsto (\phi(p), d\phi_p(X)) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}.$$

If (V, ψ) is another chart on M, then the transition map between the charts $(TU, T\phi)$ and $(TV, T\psi)$ is given by

$$T\psi \circ T\phi^{-1}(p, v) = (\psi \circ \phi^{-1}(p), D(\psi \circ \phi^{-1})(p)v).$$

Proof of Claim 4.9. Exercise.
We continue with the proof of the theorem. We endow TM with the unique maximal smooth atlas containing all the charts of the form $(TU, T\phi)$ where (U, ϕ) are charts on M, from the Claim 4.9. The second countability and Hausdorff property for TM follows from the corresponding properties of M and the fact that countable union of countable sets is again countable. Thus, TM is a $2n$-dimensional smooth manifold.

Consider the map $\pi : TM \to M$, $(p, T_pM) \mapsto p$. Consider a chart $(TU, T\phi) \ni (p, T_pM)$ with chart (U, ϕ) containing the point $\pi(p)$. Then

$$\phi \circ \pi \circ T\phi^{-1}(p, v) = \phi(p)$$

which is clearly a smooth map. Similarly, both the inclusions are smooth maps. □

Definition 4.10. The **cotangent bundle** of a manifold M is again a $2n$-dimensional smooth manifold as is defined as

$$T^*M = \bigcup_{p \in M} T_p^*M.$$

The cotangent bundle T^*M is also smooth manifold of dimension $2n$ and this can be proved in a similar manner as the case for the tangent bundle. We will learn more about the tangent and cotangent bundles later but we will keep using them.

We can now define the differential of a smooth map.

Definition 4.11. Let M and N be two smooth manifolds and let $f : M \to N$ be a smooth map. The **differential of** f, df is the map

$$df : TM \to TN : [\gamma] \mapsto [f \circ \gamma].$$

The restriction of df at every point is precisely the map df_p described in Lemma 4.4.

The proof of Lemma 4.4 also shows that if M, N and f are smooth then so is df.

Using the formulation of df described in the definition above as a map btween tangent bundles, we can now give a simole proof the very important chain rule.

Proposition 4.12 (Chain Rule). *For smooth maps $f : M \to N$, $g : N \to P$ between smooth manifolds, we have*

$$d(g \circ f) = dg \circ df : TM \to TP.$$

At every point $p \in M$, $f(p) \in N$, this reads as

$$d(g \circ f)_p = dg_{f(p)} \circ df_p. \tag{4.5}$$

Proof. We prove (4.5). Let $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, we have

$$\begin{aligned} d(g \circ f)_p(\gamma'(0)) &= \frac{d}{dt}((g \circ f) \circ \gamma)_{|t=0} \\ &= \frac{d}{dt}(g \circ (f \circ \gamma))_{|t=0} \\ &= dg_{f(p)}((f \circ \gamma)'(0)) \quad \text{(chain rule in } \mathbb{R}^n) \\ &= dg_{f(p)}(df_p(\gamma'(0))). \end{aligned}$$

□

A simple corollary of the previous proposition is the following.

Corollary 4.13. *If $f : M \to N$ is a diffeomorphism then so is $df : TM \to TN$ and moreover,*

$$(df)^{-1} = d(f^{-1}) : TN \to TM.$$

Proof. Notice that $d(id_M) : TM \to TM$ is just the map id_{TM} and hence the result follows from using the chain rule to

$$id_{TM} = d(f \circ f^{-1}) = df \circ d(f^{-1}).$$

□

We also make the following definition.

Definition 4.14. Let $k \in \mathbb{N} \cup \{\infty\}$. A surjective map $f : M \to N$ between smooth manifolds is called a **local** C^k**-diffeomorphism** if for all $p \in M$ there exists a neighbourhood $U \ni p$ and a neighbourhood $V \ni f(p)$ such that $f_U : U \to V$ is a C^k-diffeomorphism.

Clearly, every C^k-diffeomorphism is a local C^k-diffeomorphism but the opposite is not necessarily true.

Example 4.15. Let $f : \mathbb{R} \to S^1$ be the map $f(t) = e^{2\pi i t}$. Then f is a surjective map but is not injective, so it can never be a diffeomorphism. But it a local C^∞-diffeomorphism. Let $t_0 \in \mathbb{R}$ and consider $U = (t_0 - \pi, t_0 + \pi)$ and $V = S^1 \setminus \{-f(t_0)\}$.

Also notice that if f is a local diffeomorphism then $df_p : T_pM \to T_{f(p)}N$ is a linear isomorphism and as a result dim(T_pM) =dim$(T_{f(p)}M)$ and hence dim$M$ =dim(N).