2.2 Some more examples

Example 2.15. Let M^0 be a 0-dimensional manifold. We know that it a countable space with discrete topology. The charts are $\{(p, \phi) \mid p \in M, \phi : \{p\} \to \mathbb{R}^0\}$ and they trivially satisfy the compatibility conditions and thus M^0 has a unique smooth structure.

Example 2.16. The smooth structure on $\mathbb{R}^n$ determined by the atlas $(\mathbb{R}^n, id)$ is called the **standard smooth structure on $\mathbb{R}^n$.**

Example 2.17. Let V^n be a finite dimensional real vector space. Since any norm on V determines a topology and all the norms are equivalent, we use this topology to view V as a topological manifold. We show below that V is a smooth manifold. Suppose $\{e_1, \dots, e_n\}$ is an ordered basis of V and define the isomorphism $\phi : \mathbb{R}^n \to V$ by

$$\phi(x) = \sum_{i=1}^{n} x^i e_i.$$

This map is a homeomorphism and hence (V, ϕ^{-1}) is a chart.

Suppose $\{e'_1, \dots, e'_n\}$ is any other basis and $\phi'(x) = \sum_{i=1}^{n} x^i e'_i$ is the corresponding isomorphism. We know from linear algebra that there exists an invertible matrix $[A^i_j]$ such that $e_i = \sum_{j=1}^{n} A^j_i e'_j$ for each i. The transition map then is given by

$$\phi'^{-1} \circ \phi(x) = x', \quad \text{where} \quad x' \in \mathbb{R}^n \;\text{ and }\; \sum_{j=1}^{n} x^{j'} e'_j = \sum_{i=1}^{n} x^i e_i = \sum_{i,j=1}^{n} x^i A^j_i e'_j.$$

In other words, any map sending $x \mapsto x'$ is an invertible linear map and hence is a diffeomorphism, thus proving that any two charts are C^∞–compatible. The collection of all such charts define a smooth structure on V and makes it into a smooth manifold.

Example 2.18. Let $M(n \times m, \mathbb{R})$ denote the space of real $n \times m$ matrices. Since this is a real vector space of dimension nm, from the previous example, it is a smooth nm-dimensional manifold.

Example 2.19 ($\mathrm{GL}(n, \mathbb{R})$)**.** The space of $n \times n$ real invertible matrices is called the **general linear group** $\mathrm{GL}(n, \mathbb{R})$. Since this is an open subset of the set of all matrices, it is a n^2-dimensional smooth manifold.

Example 2.20. If $M_1, \dots, M_k$ are smooth manifolds of dimensions $n_1, \dots, n_k$, respectively then we already know that $M_1 \times \cdots \times M_k$ is a topological manifold of dimension $n_1 + n_2 + \cdots + n_k$. The charts are of the form $(U_1 \times \cdots \times U_k, \phi_1 \times \cdots \times \phi_k)$. We notice that if there two such charts then

$$(\psi_1 \times \cdots \times \psi_k) \circ (\phi_1 \times \cdots \times \phi_k)^{-1} = (\psi_i \circ \phi_1^{-1}) \times \cdots \times (\psi_k \circ \phi_k^{-1}),$$

making $M_1 \times \cdots \times M_k$ into a smooth manifold. For example, the n-torus T^n is a smooth manifold.

We can write a version of Proposition 1.14 for smooth manifolds as follows.

Lemma 2.21. *Let M be a set and suppose we have a collection $\{U_i\}_{i \in I}$, I is countable, of subsets of M with maps $\phi_i : U_i \to \mathbb{R}^n$ such that the following holds.*

(a) $\underset{i \in I}{U_i} = M$.

(b) For all i, ϕ_i is a bijection between U_i and an open subset $V_i \subset \mathbb{R}^n$.

(c) For all i, j, the sets $\phi_i(U_i \cap U_j)$ and $\phi_j(U_i \cap U_j)$ are open in $\mathbb{R}^n$.

(d) If $U_i \cap U_j \neq \emptyset$ for some i, j, then the map $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is smooth.

(e) For points $p, q \in M$ either there exists some $U_i \ni p, q$ or there exists distinct U_i, U_j with $p \in U_1$ and $q \in U_j$.

Then M has a unique smooth structure such that each (U_i, ϕ_i) is a smooth chart.

Proof. The proof is similar to the proof of Proposition 1.14 with the smoothness condition guaranteed by point (d) above. □

We can use the previous lemma to give an example of smooth manifolds which generalize the real projective spaces.

Example 2.22 (Grassmann manifolds). Let V^n be a real vector space and let

$$G_k(V) = \{W \subset V \mid W \text{ is a } k-\text{dimensional subspace}\}. \tag{2.5}$$

The set $G_k(V)$ is a smooth manifold of dimension $k(n-k)$ and is called the Grassmannian. Clearly, $G_1(\mathbb{R}^{n+1}) = \mathbb{RP}^n$.

We show that $G_k(V)$ is a smooth manifold. Let P and Q be complementary subspaces of V of dimensions k and $n-k$, respectively, so that $V = P \oplus Q$. If $f : P \to Q$ is a linear map then $\Gamma(f) \subset V$ is a k-dimensional subspace with

$$\Gamma(f) = \{v + fV \mid v \in P\}.$$

One can check that $\Gamma(f) \cap Q = \{0\}$. Conversely, if $S \subset V$ is any subspace with trivial intersection with Q then $S = \Gamma(f)$ where $f = (\pi_{Q_{|S}}) \circ (\pi_{P_{|S}})^{-1}$ where $\pi_T : V \to T$ is the projection.

Having established this, we let

$$L(P, Q) = \{L : P \to Q \mid L \text{ linear}\},$$

which is a vector space and let $U_Q \subset G_k(V)$ with U_Q being those k-dimensional subspaces of V which have trivial intersection with Q. Since $P \in U_Q$ so it is non-empty. The assignment $f \mapsto \Gamma(f)$ gives a map

$$\Gamma : L(P, Q) \to U_Q,$$

and is in fact, a bijection. Let us denote $\Gamma^{-1} = \phi$. Since $L(P, Q) \cong M((n-k) \times k, \mathbb{R}) \cong \mathbb{R}^{k(n-k)}$, (U_Q, ϕ) is a chart for $G_k(V)$ and part (b) of Lemma 2.21 is satisfied.

If (P', Q') is another pair of subspaces as before and $\phi' : U_{Q'} \to L(P', Q')$ is the corresponding chart, then we first of all notice that $\phi(U_Q \cap U_{Q'}) \subset L(P, Q)$ is the set of all linear maps $f : P \to Q$ such that $\Gamma(f) \cap Q' = \{0\}$. This set $\phi(U_Q \cap U_{Q'})$ is open in $L(P, Q)$. This can be seen as follows. Let $f \in \phi(U_Q \cap U_{Q'})$ and this means that $\Gamma(f) \cap Q' = \{0\}$. If we denote by $I_f : P \to V$, $I_f(p) = p + fv$ then it is abijection from P to $\Gamma(f)$. Moreover, $\ker \pi_{P'} = Q'$ so using the linear algebraic fact that

$$\text{rank}(A \circ B) \leq \text{rank}(B) \text{ with equality } \iff \text{Im}(B) \cap \ker A = \{0\},$$

we see that $\pi_{P'} \circ I_f$ has full rank and since the corresponding matrix depend continuously on f, we see that the set of all such f with $\Gamma(f) \cap Q' = \{0\}$ is open. Thus, part (c) of Lemma 2.21 is also true.

We now show that the transition maps are smooth and leave the rest of the details as an exercise. We want to show that $\phi' \circ \phi^{-1}$ is a smooth map on $\phi(U_Q \cap U_{Q'})$. Let $f \in \phi(U_Q \cap U_{Q'}) \subset L(P, Q)$ be arbitrary. Let S denote the subspace $\Gamma(f) \subset V$. If $f' = \phi' \circ \phi^{-1}(f)$, then, $f' = (\pi_{Q'_{|S}}) \circ (\pi_{P'_{|S}})^{-1}$. Somehow, we would like to relate this map to f. This can be done as follows: note that $I_f : P \to S = \Gamma(f)$ is an isomorphism so

$$f' = (\pi_{Q'_{|S}}) \circ I_f \circ (I_f)^{-1} \circ (\pi_{P'_{|S}})^{-1} = (\pi_{Q'} \circ I_f) \circ (\pi_{P'} \circ I_f)^{-1}.$$

Now we can choose any basis for V and write a basis for spaces $P, P', Q.Q'$ and then let $A : P \to P'$, $B : P \to Q'$, $C : Q \to P'$ and $D : Q \to Q'$ be the linear maps with

$$A = \pi_{P'_{|P}}, \ B = \pi_{Q'_{|P}}, \ C = \pi_{P'_{|Q}}, \text{ and } D = \pi_{Q'_{|Q}}.$$

So, for instance, for $p \in P$, we get

$$(\pi_{P'} \circ I_f)p = (A + Cf)p, \ (\pi_{Q'} \circ I_f)p = (B + Df)p,$$

and hence $f' = (B + Df)(A + Cf)^{-1}$. But then we are done because all these are matrices whose entries are just rational functions depending on f and as a result, the matrix entries of f' depend smoothly on those of f. Since f was smooth then so is the transition map. Thus, part (d) of Lemma 2.21 is also satisfied.

Part (a) also follows because if say, $\{e_1, \ldots, e_n\}$ is any basis of V and $F \subset \{1, \ldots, n\}$ with $|F| = k$. Then if $V_F = \text{linear span}\{e_1, i \in F\}$ and we let $U_F = U_{V_F}$ with corresponding $\phi_{V_F} = \phi_F$ then the finite collection $\{(U_F, \phi_F) \mid F \subset \{1, \ldots, n\}, \ |F| = k\}$ is a smooth atlas of $G_k(V)$. Similarly, one can check that $G_k(V)$ is indeed Hausdorff and hence the Grassmannian is a smooth manifold.

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