16. Stokes's theorem

16.1 Orientations on manifolds with boundary

We start with our discussion on manifold with boundary. It's almost the same concept as that of a manifold but now we allow points in the "boundary" as well which will be described as those points such that any neighbourhood looks like a part of "half-space". Let

$$\mathbb{R}^m_+ = \{(x^1, \dots, x^m) \mid x^m \geq 0\}.$$

We have the following.

Definition 16.1. A topological space M is called an m-dimensional **topological manifold with boundary** if it is Hausdorff, second-countable, and for any $p \in M$, there is a neighborhood U of p which is homeomorphic to either $\mathbb{R}^m$ or $\mathbb{R}^m_+$.
We define the boundary of M to be

$$\partial M = \{p \in M \mid p \text{ has no neighborhood in } M \text{ that is homeomorphic to } \mathbb{R}^m\}.$$

Thus, $p \in \partial M$ if and only if p has a neighborhood that is homeomorphic to $\mathbb{R}^m_+$ such that p gets mapped onto $\partial \mathbb{R}^m_+ = \{(x^1, \dots, x^m) \mid x^m = 0\}$. We call

$$\text{int}(M) = M \setminus \partial M$$

the interior of M.

Example 16.2. The closed ball

$$B^m(1) = \{x \in \mathbb{R}^m \mid |x| \leq 1\}$$

is a smooth manifold with boundary, and its boundary is $\partial B^m(1) = S^{m-1}$.

Lemma 16.3. *If M^m is a smooth manifold then the smooth structure on M induces a smooth structure on ∂M making it an $(m-1)$-dimensional smooth manifold.*

Proof. Let $p \in \partial M$ and let $(U, x^1, \dots, x^m)$ be a chart coming from the smooth structure on M which is homeomorphic to an open subset of $\mathbb{R}^m_+$. Then

$$U \cap \partial M = \{(x^1, \dots, x^m) \mid x^m = 0\},$$

and hence $(U \cap \partial M, x^1, \dots x^{m-1})$ is a chart on ∂M. □

Now let M be an orientable smooth manifold with boundary, and let $[\mu]$ be an orientation on M. This means that $\mu \in \Omega^m(M)$ is nowhere vanishing. We obtain an induced orientation on ∂M. The induced orientation on ∂M is defined as follows. Suppose locally near the boundary, we have

$$\mu = f dx^1 \wedge \cdots \wedge dx^m$$

with $f > 0$. Then, locally, we define the induced orientation on ∂M to be the one that is represented by the differential form

$$\eta = (-1)^m dx^1 \wedge \cdots \wedge dx^{m-1}. \tag{16.1}$$

By choosing a partitions of unity, we can glue these local $(m-1)$-forms on ∂M into a global $(m-1)$-form on ∂M which is nowhere vanishing. The nowhere vanishing property of η follows from the nowhere vanishing property of μ. Thus, from Theorem 15.14, we get a volume form that defines an orientation on ∂M, which is induced from an orientation on M.

We explain why η in (16.1) has that form. Note that in coordinate charts near boundary, M is defined by $x^m \geq 0$. So $-x^m$ is the "outward pointing normal" of M, and this boundary orientation is chosen so that

$$d(-x^m) \wedge \eta = d(-x^m) \wedge (-1)^m dx^1 \wedge \cdots \wedge dx^{m-1} = dx^1 \wedge \cdots \wedge dx^m.$$

16.2 Stokes's theorem

Now we can state and prove our main theorem:

Theorem 16.4 (Stokes's theorem). *Let M be a smooth oriented m-dimensional manifold with boundary ∂M with the induced orientation. For any $\omega \in \Omega^{m-1}(M)$ with compact support, we have*

$$\int_{\partial M} \iota_{\partial M}^* \omega = \int_M d\omega, \tag{16.2}$$

where $\iota_{\partial M} : \partial M \hookrightarrow M$ is the inclusion map.

Proof. The proof follows from the following three cases which we prove one by one.
(1) ω is supported in a chart U that is diffeomorphic to $\mathbb{R}^m$.
(2) ω is supported in a chart U that is diffeomorphic to $\mathbb{R}^m_+$.
(3) The general case.

Case (1): Since $\omega = 0$ on ∂M, we have $\int_{\partial M} \iota_{\partial M}^* \omega = 0$. To calculate $\int_M d\omega$, we denote

$$\omega = \sum_i (-1)^{i-1} f_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^m,$$

where f_i's are compactly supported smooth functions. Then, using the formula for the exterior derivative, we have

$$d\omega = \sum_i \frac{\partial f_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^m.$$

So by definition,

$$\begin{aligned}
\int_M d\omega &= \int_{\mathbb{R}^m} \sum_i \frac{\partial f_i}{\partial x^i} dx^1 \cdots dx^m \\
&\underbrace{=}_{\text{Fubini's theorem}} \sum_i \int_{\mathbb{R}^{m-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x^i} dx^i \right) dx^1 \cdots \widehat{dx^i} \cdots dx^m \\
&\underbrace{=}_{\text{FTC}} \sum_i \int_{\mathbb{R}^{m-1}} \left(\lim_{s \to \infty} f_i(s) - \lim_{s \to -\infty} f_i(s) \right) dx^1 \cdots \widehat{dx^i} \cdots dx^m \\
&= 0,
\end{aligned}$$

as f is compactly supported. Thus, (16.2) is true in Case 1.

Case (2): We have the same formula to calculate $\int_M d\omega$, except for the $i = m$ term, because in this case one of the limits is 0 instead of $-\infty$. We have

$$\int_M d\omega = \int_{\mathbb{R}^{m-1}} \left(\int_0^{\infty} \frac{\partial f_m}{\partial x^m} dx^m \right) dx^1 \cdots dx^{m-1} = -\int_{\mathbb{R}^{m-1}} f_m(x^1, \ldots, x^{m-1}, 0) dx^1 \cdots dx^{m-1}.$$

On the other hand, since $x^m = 0$ on ∂M, we see

$$\iota_{\partial M}^* \omega = (-1)^{m-1} f_m(x^1, \ldots, x^{m-1}, 0) dx^1 \wedge \cdots \wedge dx^{m-1} = -f_m(x^1, \ldots, x^{m-1}, 0) \cdot (-1)^m dx^1 \wedge \cdots \wedge dx^{m-1}.$$

So

$$\begin{aligned}
\int_{\partial M} \iota_{\partial M}^* \omega &= \int_{\mathbb{R}^{m-1}} (-f_m(x^1, \ldots, x^{m-1}, 0)) dx^1 \cdots dx^{m-1} \\
&= -\int_{\mathbb{R}^{m-1}} f_m(x^1, \ldots, x^{m-1}, 0) dx^1 \cdots dx^{m-1} \\
&= \int_M d\omega,
\end{aligned}$$

which is (16.2).

Case (3): In general, we cover the compact set supp(ω) by finitely many coordinate charts, and take a partition of unity $\{\psi_i\}$ subordinate to that cover. Then

$$\int_{\partial M} \iota_{\partial M}^* \omega = \sum_i \int_{\partial M} \iota_{\partial M}^* (\psi_i \omega) \underbrace{=}_{\text{Stokes's thm. in a chart}} \sum_i \int_M d(\psi_i \omega) = \sum_i \int_M d\psi_i \wedge \omega + \int_M d\omega = \int_M d\omega,$$

as

$$\sum_i \int_M d\psi_i \wedge \omega = \int_M d(\sum_i \psi_i) \wedge \omega = 0.$$

□

We see some simple consequences.

1. Let $M = [a, b]$ and $\omega = f \in C^\infty(M) = \Omega^0(M)$. Then

$$\int_M df = \int_a^b \frac{df}{dx} dx = \int_{\partial[a,b]} f = f(b) - f(a).$$

2. If $\partial M = \emptyset$ then for any compactly supported $m-1$-form ω we have $\int_M d\omega = \int_{\partial M} \omega = 0$. For instance, if $f : S^1 \to \mathbb{R}$ is a smooth function then $\int_{S^1} df = 0$.

3. **Green's theorem** Let D be a compact domain in $\mathbb{R}^2$ and $P, Q \in C^\infty(D)$. Then

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dx dy \right) = \int_{\partial D} P dx + Q dy.$$