14.4 Integrations of top forms on smooth manifolds.

Now assume M is a smooth orientable m-manifold and fix an orientation $\mathcal{A}$ on M. Let ω be any smooth m-form on M. To define $\int_M \omega$, we first assume that ω is supported in a coordinate chart (φ, U, V) which is orientation compatible with $\mathcal{A}$. In this case there is a function f supported in U such that

$$\omega = f(\varphi(x))dx^1 \wedge \ldots \wedge dx^m.$$

In this case we simply define

$$\int_U \omega := \int_V f(x)dx^1 \ldots dx^m, \tag{14.5}$$

where the right hand side is the Lebesgue integral on $V \subset \mathbb{R}^m$. We will assume f is integrable. In fact, in what follows the function f involved are compactly supported and hence integrable.

To integrate a general m-form $\omega \in \Omega^m(M)$, we take a locally finite cover $\{U_\alpha\}$ of M that are compatible with the orientation $\mathcal{A}$. Let $\{\psi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Now since each ψ_α is supported in U_α, each $\psi_\alpha\omega$ is supported in U_α as well. We define

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} \psi_\alpha \omega. \tag{14.6}$$

We say that ω is **integrable** if the right hand side converges absolutely for any such cover and any such partitions of unity. This is true, for example, if ω is compactly supported.

One need to check that the definition (14.6) is independent of the choices of orientation-compatible coordinate charts, and is independent of the choices of partition of unity.

Theorem 14.10. *Suppose ω is compactly supported, or more generally, ω is integrable. The expression (14.6) is independent of the choices of $\{U_\alpha\}$ and the choices of $\{\psi_\alpha\}$.*

Proof. We first show that (14.5) is well-defined. The argument is essentially the same as in the Euclidean case: if ω is supported in U, and if $\{x_\alpha^i\}$ and $\{x_\beta^j\}$ are two orientation-compatible coordinate systems on U, so that

$$\omega = f_\alpha dx_\alpha^1 \wedge \ldots \wedge dx_\alpha^m = f_\beta dx_\beta^1 \wedge \ldots \wedge dx_\beta^m,$$

then we want to prove

$$\int_{V_\alpha} f_\alpha dx_\alpha^1 \ldots dx_\alpha^m = \int_{V_\beta} f_\beta dx_\beta^1 \ldots dx_\beta^m.$$

This is true, because

$$dx_\beta^1 \wedge \ldots \wedge dx_\beta^m = \det(d\varphi_{\alpha\beta}) dx_\alpha^1 \wedge \ldots \wedge dx_\alpha^m$$

implies $f_\alpha = \det(d\varphi_{\alpha\beta}) f_\beta$. Since $\det(d\varphi_{\alpha\beta}) > 0$, the conclusion follows from the change of variable formula in $\mathbb{R}^n$.

To prove (14.6) is well-defined, we suppose $\{U_\alpha\}$ and $\{U_\beta\}$ are two locally finite cover of M consisting of orientation-compatible charts, and $\{\psi_\alpha\}$ and $\{\rho_\beta\}$ are partitions of unity subordinate to $\{U_\alpha\}$ and $\{U_\beta\}$ respectively. Then $\{U_\alpha \cap U_\beta\}$ is a new locally finite cover of M, and $\{\psi_\alpha \rho_\beta\}$ is a partition of unity subordinate to this new cover. It is enough to prove

$$\sum_\alpha \int_{U_\alpha} \psi_\alpha \omega = \sum_{\alpha,\beta} \int_{U_\alpha \cap U_\beta} \psi_\alpha \rho_\beta \omega.$$

This is true because for each fixed α,

$$\int_{U_\alpha} \psi_\alpha \omega = \int_{U_\alpha} \Big(\sum_\beta \rho_\beta\Big) \psi_\alpha \omega = \sum_\beta \int_{U_\alpha \cap U_\beta} \psi_\alpha \rho_\beta \omega.$$

□

14.5 Change of variable formula

Finally we extend the change of variable formula form $\mathbb{R}^m$ to manifolds.

Definition 14.11. Let M, N be orientable smooth n-manifolds, with orientations $\mathcal{A}$ and $\mathcal{B}$ respectively. A diffeomorphism $\varphi : M \to N$ is said to be **orientation-preserving** if for each $(\psi_\beta, X_\beta, Y_\beta) \in \mathcal{B}$, the chart $(\psi_\beta \circ \varphi, \varphi^{-1}(X_\beta), Y_\beta)$ on M is orientation compatible with $\mathcal{A}$.

Suppose M, N are connected. It is easy to see that a diffeomorphism $\varphi : M \to N$ is orientation-preserving if and only if there exists one chart $(\psi_\beta, X_\beta, Y_\beta) \in \mathcal{B}$, such that the chart $(\psi_\beta \circ \varphi, \varphi^{-1}(X_\beta), Y_\beta)$ on M is orientation compatible with $\mathcal{A}$. Similarly if there exists one chart $(\psi_\beta, X_\beta, Y_\beta) \in \mathcal{B}$, such that the chart $(\psi_\beta \circ \varphi, \varphi^{-1}(X_\beta), Y_\beta)$ on M is incompatible with $\mathcal{A}$, then for every chart $(\psi_\beta, X_\beta, Y_\beta) \in \mathcal{B}$, the chart $(\psi_\beta \circ \varphi, \varphi^{-1}(X_\beta), Y_\beta)$ on M is incompatible with $\mathcal{A}$. In this case we say φ is *orientation-reverting*.

Now we state:

Theorem 14.12 (The change of variable formula)**.** *Suppose M, N are n-dimensional orientable smooth manifolds, and $\varphi : M \to N$ is a diffeomorphism.*

1. *If φ is an orientation-preserving, then*
$$\int_M \varphi^*\omega = \int_N \omega.$$
2. *If φ is an orientation-reverting, then*
$$\int_M \varphi^*\omega = -\int_N \omega.$$

Proof. It is enough to prove this in local charts, in which case this is merely the change of variable formula in $\mathbb{R}^m$. □

Remark 14.13. If ω is a compactly supported k-form on M, where $k < m = \dim M$, then one cannot integrate ω over M. However, for any k-dimensional orientable submanifold $X \subset M$, one can define $\int_X \omega$ by setting it to be $\int_X \iota^*\omega$, where $\iota : X \hookrightarrow M$ is the inclusion map. By this way we get a "pairing" between k-forms on M and k-dimensional orientable submanifolds in M.

14.6 Volume form and volume measure

Next we show that orientability can be characterized via the existence of specific top forms:

Theorem 14.14. *An m-dimensional smooth manifold M is orientable if and only if M admits a nowhere vanishing smooth m-form μ.*

Proof. First let μ be a nowhere vanishing smooth m-form on M. Then on each local chart $(U, x^1, \ldots, x^m)$ (where U is always chosen to be connected), there is a smooth function $f \neq 0$ so that $\mu = f dx^1 \wedge \ldots \wedge dx^m$. It follows that
$$\mu(\partial_1, \ldots, \partial_m) = f \neq 0.$$
We can always take such a chart near each point so that $f > 0$, otherwise we can replace x^1 by $-x^1$. Now suppose $(U_\alpha, x_\alpha^1, \ldots, x_\alpha^m)$ and $(U_\beta, x_\beta^1, \ldots, x_\beta^m)$ be two such charts, so that on the intersection $U_\alpha \cap U_\beta$ one has
$$f dx_\alpha^1 \wedge \ldots \wedge dx_\alpha^m = \mu = g dx_\beta^1 \wedge \ldots \wedge dx_\beta^m = g \det(d\varphi_{\alpha\beta}) dx_\alpha^1 \wedge \ldots \wedge dx_\alpha^m.$$
where $f, g > 0$. It follows that $\det(d\varphi_{\alpha\beta}) > 0$. So the atlas constructed by this way is an orientation.

Conversely, suppose $\mathcal{A}$ is an orientation. For each local chart U_α in $\mathcal{A}$, we let
$$\mu_\alpha = dx_\alpha^1 \wedge \ldots \wedge dx_\alpha^m.$$

Pick a partition of unity $\{\psi_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$. We claim that

$$\mu := \sum_\alpha \psi_\alpha \mu_\alpha,$$

is a nowhere vanishing smooth m-form on M. In fact, for each $p \in M$, there is a neighbourhood U of p so that the sum $\sum_\alpha \psi_\alpha \mu_\alpha$ is a finite sum $\sum_{i=1}^k \psi_i \mu_i$. It follows that near p,

$$\mu(\partial_1^1, \dots, \partial_m^1) = \sum_{i=1}^k (\det d\varphi_{1i})\, \psi_i > 0.$$

So $\mu \neq 0$ near p. □

Definition 14.15. A nowhere vanishing smooth m-form μ on an m-dimensional smooth manifold M is called a *volume form*.

Thus, we can always integrate top forms and its multiples on manifolds admitting volume forms.

Remark 14.16. If M is orientable, and μ is a volume form, then the two orientations of M are represented by μ and $-\mu$ respectively. We denote the two orientations by $[\mu]$ and $[-\mu]$.

Remark 14.17. In particular, on any Lie group, one can define conceptions like left-invariant differential forms. Since any Lie group is orientable (Exercise), there exists left-invariant volume form on any Lie group G. The measures associated to left-invariant volume form on Lie groups are called **Haar measures**.