14. Integration on manifolds

14.1 Partitions of unity

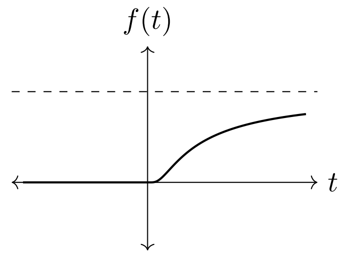
We start our discussions with one of the most important classes of smooth functions called **bump functions**. Recall that we used these functions in our proof of the Tensor characterization lemma. We start with the following fact from real analysis.

Proposition 14.1. *The function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(t) = \begin{cases} e^{-1/t}, & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{14.1}$$

is smooth. □

The function's graph look like the figure below.



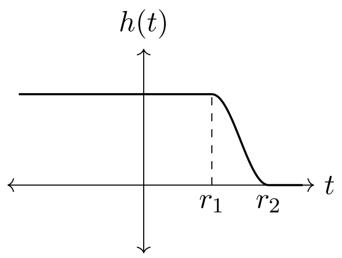
As a result, given any real numbers r_1 and r_2 such that $r_1 < r_2$, there exists a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} h(t) \equiv 1, & t \leq r_1, \\ 0 < h(t) < 1, & r_1 < t < r_2, \\ h(t) \equiv 0, & t \geq r_2 \end{cases}$$

by defining

$$h(t) = \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}.$$

with $f(t)$ as in (14.1). Such an h is called a **cutoff function** and it looks like the figure below.



We can easily extend this construction to $\mathbb{R}^n$ as the following result shows.

Proposition 14.2. *Given any positive real numbers $r_1 < r_2$, there is a smooth function $H : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\begin{cases} H(x) \equiv 1, & x \in \overline{B}_{r_1}(0) \\ 0 < H(x) < 1, & x \in B_{r_2}(0) \setminus \overline{B}_{r_1}(0) \\ H(x) \equiv 0, x \in \mathbb{R}^n \setminus B_{r_2}(0). \end{cases} \tag{14.2}$$

Proof. Set $H(x) = h(|x|)$, with h as defined above. Clearly, H is smooth on $\mathbb{R}^n \setminus \{0\}$, because it is a composition of smooth functions there. Since it is identically equal to 1 on $B_{r_1}(0)$, it is smooth there too. $\square$

The function H constructed in this proposition is an example of a **smooth bump function**, a smooth real-valued function that is equal to 1 on a specified set and is zero outside a specified neighbourhood of that set.

We can now define partitions of unity.

Definition 14.3. Let M be a manifold and let $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ be an arbitrary open cover of M. A **partition of unity subordinate to** $\mathcal{U}$ is an indexed family $(\psi_\alpha)_{\alpha\in A}$ of continuous functions $\psi_\alpha : M \to \mathbb{R}$ with the following properties:

(i) $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in A$ and all $x \in M$.

(ii) $\operatorname{supp} \psi_\alpha \subseteq U_\alpha$ for each $\alpha \in A$.

(iii) The family of supports $(\operatorname{supp} \psi_\alpha)_{\alpha\in A}$ is locally finite, meaning that every point has a neighborhood that intersects $\operatorname{supp} \psi_\alpha$ for only finitely many values of α.

(iv) $\sum_{\alpha\in A} \psi_\alpha(x) = 1$ for all $x \in M$.

If M is a smooth manifold, a **smooth partition of unity** is one for which each of the functions ψ_α is smooth.

Because of the local finiteness condition (iii), the sum in (iv) actually has only finitely many nonzero terms in a neighbourhood of each point, so there is no issue of convergence.

Theorem 14.4. *Suppose M is a smooth manifold and $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ is any indexed open cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{U}$.*

Proof. Each set U_α is a smooth manifold in its own right, and thus has a basis $\mathcal{B}_\alpha$ of coordinate balls and thus $\mathcal{B} = \bigcup_\alpha \mathcal{B}_\alpha$ is a basis for the topology of M. Since M is a manifold, the cover $\mathcal{U}$ has a countable, locally finite refinement $\{B_i\}$ consisting of elements of $\mathcal{B}$ and the cover $\{\overline{B}_i\}$ is also locally finite.

For each i, the fact that B_i is a coordinate ball in some U_α implies that there is a coordinate ball $B_i' \subseteq U_\alpha$ such that $B_i' \supseteq \overline{B}_i$, and a smooth coordinate map $\varphi_i : B_i' \to \mathbb{R}^n$ such that $\varphi_i(\overline{B}_i) = \overline{B}_{r_i}(0)$ and $\varphi_i(B_i') = B_{r_i'}(0)$ for some $r_i < r_i'$. For each i, define a function $f_i : M \to \mathbb{R}$ by

$$f_i = \begin{cases} H_i \circ \varphi_i & \text{on } B_i', \\ 0 & \text{on } M \setminus \overline{B}_i, \end{cases}$$

where $H_i : \mathbb{R}^n \to \mathbb{R}$ is a smooth function that is positive in $B_{r_i}(0)$ and zero elsewhere, as in Proposition 14.2. On the set $B_i' \setminus \overline{B}_i$ where the two definitions overlap, both definitions yield the zero function, so f_i is well defined and smooth, and $\operatorname{supp} f_i = \overline{B}_i$.

Define $f : M \to \mathbb{R}$ by

$$f(x) = \sum_i f_i(x).$$

Because of the local finiteness of the cover $\{\overline{B}_i\}$, this sum has only finitely many nonzero terms in a neighbourhood of each point and thus defines a smooth function. Because each f_i is non-negative everywhere and positive on B_i, and every point of M is in some B_i, it follows that $f(x) > 0$ everywhere on M. Thus, the functions $g_i : M \to \mathbb{R}$ defined by $g_i(x) = f_i(x)/f(x)$ are also smooth. It is immediate from the definition that $0 \leq g_i \leq 1$ and $\sum_i g_i \equiv 1$.

Finally, we need to re-index our functions so that they are indexed by the same set A as our open cover. Because the cover $\{B_i'\}$ is a refinement of $\mathcal{U}$, for each i we can choose some index $a(i) \in A$ such that $B_i' \subseteq U_{a(i)}$. For each $\alpha \in A$, define $\psi_\alpha : M \to \mathbb{R}$ by

$$\psi_\alpha = \sum_{i : a(i) = \alpha} g_i.$$

If there are no indices i for which $a(i) = \alpha$, then this sum should be interpreted as the zero function. Thus,

$$\operatorname{supp} \psi_\alpha = \overline{\bigcup_{i:a(i)=\alpha} B_i} = \bigcup_{i:a(i)=\alpha} \overline{B}_i \subseteq U_\alpha.$$

Each ψ_α is a smooth function that satisfies $0 \leq \psi_\alpha \leq 1$. Moreover, the family of supports $(\operatorname{supp} \psi_\alpha)_{\alpha \in A}$ is still locally finite, and $\sum_\alpha \psi_\alpha \equiv \sum_i g_i \equiv 1$, so this is the desired partition of unity. $\square$

14.2 Top forms on manifolds

Let M^n be a smooth manifold. Since, $\Omega^k(M) = 0$ for $k > m$, we call any smooth n-form a **top form** on M. Let $p \in M$ and (φ, U) a local chart near p. Then $dx^1 \wedge \cdots \wedge dx^n$ is a top form on U. Note that for any $q \in U$, $(dx^1 \wedge \ldots \wedge dx^m)_q \neq 0$ since at any $q \in U$,

$$(dx^1 \wedge \cdots \wedge dx^n)_q(\partial_1, \ldots, \partial_n) = \det(dx^i(\partial_j))_{1 \leq i,j \leq n} = 1.$$

Moreover, since $\dim \Lambda^n T_p^* M = 1$, we see that for any top form ω on U and any $q \in U$, there is a real number λ_q such that

$$\omega_q = \lambda_q (dx^1 \wedge \ldots \wedge dx^m)_q.$$

Since ω is C^∞, the coefficient λ, as a function on U, is C^∞. So up to multiplication by functions, there is a "canonical top form" in the chart. Of course, this may not be true globally on M: For two top forms $\omega, \eta \in \Omega^n(M)$, it may happen that $\omega_p = 0, \omega_q \neq 0$ while $\eta_p \neq 0, \eta_q = 0$, and thus there exists no function $f \in C^\infty(M)$ with $\omega = f\eta$ or $\eta = f\omega$.

In particular, if we change coordinates from $(x_\alpha^1, \ldots, x_\alpha^n)$ to $(x_\beta^1, \ldots, x_\beta^n)$ on U, we get two top forms, $dx_\alpha^1 \wedge \ldots \wedge dx_\alpha^n$ and $dx_\beta^1 \wedge \ldots \wedge dx_\beta^n$. They should be related by a smooth function on U.

Proposition 14.5. *If $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism and $y = \varphi(x)$, then*

$$\varphi^*(dy^1 \wedge \ldots \wedge dy^n) = \det(d\varphi_x)\, dx^1 \wedge \ldots \wedge dx^n$$

Proof. If we denote $\varphi = (\varphi^1, \ldots, \varphi^n)$, then $\varphi^* y^i = y^i \circ \varphi = \varphi^i$. So

$$\varphi^*(dy^1 \wedge \ldots \wedge dy^n) = d\varphi^1 \wedge \ldots \wedge d\varphi^n.$$

But since

$$d\varphi^1 \wedge \ldots \wedge d\varphi^m(\partial_1^x, \ldots, \partial_n^x) = \det(d\varphi_x),$$

we have

$$d\varphi^1 \wedge \ldots \wedge d\varphi^n = \det(d\varphi_x)\, dx^1 \wedge \ldots \wedge dx^n.$$

$\square$

Now let $(\varphi_\alpha, U, V_\alpha)$ and $(\varphi_\beta, U, V_\beta)$ be two coordinate charts. Then the coordinate change map is $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$, which maps $\varphi_\alpha(x)$ to $y = \varphi_\beta(x)$. So we get

$$(\varphi_{\alpha\beta})^* (\varphi_\beta^{-1})^* dx_\beta^1 \wedge \ldots \wedge dx_\beta^n = \det(d\varphi_{\alpha\beta})\; (\varphi_\alpha^{-1})^*\; dx_\alpha^1 \wedge \ldots \wedge dx_\alpha^n.$$

Since $(\varphi_{\alpha\beta})^*(\varphi_\beta^{-1})^* = (\varphi_\beta^{-1} \circ \varphi_{\alpha\beta})^* = \varphi_\alpha^{-1*}$, we get at

$$dx_\beta^1 \wedge \ldots \wedge dx_\beta^n = \det(d\varphi_{\alpha\beta})\; dx_\alpha^1 \wedge \ldots \wedge dx_\alpha^n.$$

14.3 Orientability

Let M be a smooth manifold of dimension m, and let $\omega \in \Omega^m(M)$ be a smooth m-form. We want to define the integral $\int_M \omega$. For simplicity let's suppose ω is supported on a chart (φ, U, V) with coordinates $\{x^1, \dots, x^m\}$. Then we can write

$$\omega = f(\varphi(x))dx^1 \wedge \dots \wedge dx^m,$$

where f is a smooth function on V. With the help of the Euclidean differential form $f(x)dx^1 \dots dx^m$ on V, it is natural to define

$$\int_U \omega := \int_V f(x)dx^1 \dots dx^m. \tag{14.3}$$

Then as usual one need to check that the integral in the right hand side is independent of the choice of coordinate charts.

So we let $(\varphi_\alpha, U, V_\alpha)$ and $(\varphi_\beta, U, V_\beta)$ be two coordinate systems on U, with transition map $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : V_\alpha \to V_\beta$ which maps $x_\alpha = \varphi_\alpha(x)$ to $x_\beta = \varphi_\beta(x)$. Then

$$\omega = f_\beta(x_\beta)dx_\beta^1 \wedge \dots \wedge dx_\beta^m = f_\beta(\varphi_{\alpha\beta}(x_\alpha)) \det(d\varphi_{\alpha\beta})dx_\alpha^1 \wedge \dots \wedge dx_\alpha^m$$

So for the definition to be well-defined, we need

$$\int_{V_\beta} f_\beta(x_\beta)dx_\beta^1 \dots dx_\beta^m = \int_{V_\alpha} f_\beta(\varphi_{\alpha\beta}(x_\alpha)) \det(d\varphi_{\alpha\beta})dx_\alpha^1 \dots dx_\alpha^m.$$

Unfortunately this is not always true: In calculus we learned that for the integrals of multi-variable functions

$$\int f(x)dx^1 \dots dx^m,$$

if $\varphi : V_1 \to V_2$ is a diffeomorphism, then we have the *change of variable formula*: $(y = \varphi(x))$

$$\int_{V_2} f(x)dy^1 \dots dy^m = \int_{V_1} f(\varphi(x))|\det(d\varphi)(x)|dx^1 \dots dx^m. \tag{14.4}$$

So in order to have

$$\int_{V_\beta} f_\beta(x_\beta)dx_\beta^1 \dots dx_\beta^m = \int_{V_\alpha} f_\beta(\varphi_{\alpha\beta}(x_\alpha))\ \det(d\varphi_{\alpha\beta})(x_\alpha)\ dx_\alpha^1 \dots dx_\alpha^m.$$

and thus, for the definition to be independent of the choice of charts, we need to assume

$$\det(d\varphi_{\alpha\beta}) > 0$$

for all charts. This motivates the following definition.

Definition 14.6. Let M be a smooth manifold of dimension n. Two charts $(\varphi_\alpha, U_\alpha, V_\alpha)$ and $(\varphi_\beta, U_\beta, V_\beta)$ are **orientation compatible** if the transition map $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ satisfies

$$\det(d\varphi_{\alpha\beta})_p > 0,$$

for all $p \in \varphi_\alpha(U_\alpha \cap U_\beta)$.
An **orientation** of M is an atlas $\mathcal{A} = \{(\varphi_\alpha, U_\alpha, V_\alpha) \mid \alpha \in A\}$ whose charts are pairwise orientation compatible.
We say M is **orientable** if it has an orientation.

Remark 14.7. Let U be a chart with coordinates $\{x^1, \dots, x^m\}$. We use the notation $-U$ to represent the same coordinate chart U but with "twisted" coordinates $\{-x^1, x^2, \dots, x^m\}$. Then $-U$ and U are not orientation compatible. Let $\tilde{U}$ be any other coordinate chart such that $\tilde{U} \cap U \neq \emptyset$ is *connected*. Then either

- $\tilde{U}$ and U are orientation compatible,

or

- $\tilde{U}$ and $-U$ are orientation compatible.

As a consequence, we immediately see

Corollary 14.8. *If M is connected and orientable, then M admits exactly two different orientations.*

Example 14.9. For the real projective space $\mathbb{RP}^n$, we have constructed an atlas consisting of $n+1$ charts. We have seen that $\mathbb{RP}^n$ is orientable for odd n. It turns out that $\mathbb{RP}^n$ is not orientable for even n.