Proof. Define $d\omega$ via the formula in (13.19). We need to show that

1. $d\omega$ is an alternating tensor, i.e. for any $m < n$,

$$d\omega(X_1, \ldots, X_m, \ldots, X_n, \ldots, X_{k+1}) = -d\omega(X_1, \ldots, X_n, \ldots, X_m, \ldots, X_{k+1}).$$

This can be checked to be true from (13.19).

(2) $d\omega$ is multi-linear is $C^\infty(U)$-linear on U. Note that $d\omega$ is obviously $\mathbb{R}$-linear. We prove that for any $f \in C^\infty(U)$,

$$d\omega(fX_1, X_2, \ldots, X_{k+1}) = f d\omega(X_1, \ldots, X_{k+1}).$$

Let use compute.

$$\begin{aligned} d\omega(fX_1, X_2, \ldots, X_{k+1}) =& fX_1(\omega(X_2, \ldots, X_{k+1})) \\ &+ \sum_{i>1} (-1)^{i-1} X_i(\omega(fX_1, \ldots, \widehat{X}_i, \ldots, X_{k+1})) \\ &+ \sum_{i>1} (-1)^{1+i} \omega([fX_1, X_i], X_2, \ldots, \widehat{X}_i, \ldots, X_{k+1}) \\ &+ \sum_{1<i<j} (-1)^{i+j} \omega([X_i, X_j], fX_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1}) \\ =& f d\omega(X_1, \ldots, X_{k+1}) \\ &+ \sum_{i>1} (-1)^{i-1} (X_i f) \omega(X_1, \ldots, \widehat{X}_i, \ldots, X_{k+1}) \\ &- \sum_{i>1} (-1)^{1+i} (X_i f) \omega(X_1, \ldots, \widehat{X}_i, \ldots, X_{k+1}) \\ =& f d\omega(X_1, \ldots, X_{k+1}), \end{aligned}$$

where we used the property $([fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X)$ in the second to last equality.

(3) We now check that $d\omega$ has the local expression (13.17). This is an exercise.

□

We get the same formulas for wedge product, interior product and pull-backs of differential forms on manifolds as we had them for forms on vector spaces.

The *wedge product* $\wedge : \Omega^k(U) \times \Omega^l(U) \to \Omega^{k+l}(U)$.
For example,

$$(dx^1 + 2dx^2) \wedge (dx^1 \wedge dx^2 - dx^2 \wedge dx^3 + 3dx^1 \wedge dx^3) = -7dx^1 \wedge dx^2 \wedge dx^3.$$

For any $V \in \Gamma(TM)$ we have the *interior product* $i_V : \Omega^k(M) \to \Omega^{k-1}(M)$ with

$$V \lrcorner \omega(X_1, \ldots, X_{k-1}) = i_V \omega(X_1, \ldots, X_{k-1}) = \omega(V, X_1, \ldots, X_{k-1}).$$

For any smooth map $F : M \to N$, one has the *pull-back* $F^* : \Omega^k(N) \to \Omega^k(M)$. This is defined pointwise via the linear map $dF_p : T_pM \to T_{F(p)}N$. So if $\omega \in \Omega^k(N)$, then

$$(F^*\omega)_p(X_1, \ldots, X_k) = \omega_{F(p)}(dF_p(X_1), \ldots, dF_p(X_k)).$$

We list all the properties related to differential forms on manifolds.

Lemma 13.18. *Let M and N be a smooth manifolds. Suppose $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$, $X \in \Gamma(TM)$ and $F : M \to N$ be a smooth map. Then*

1. $\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega.$
2. $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta.$
3. $i_X(\omega \wedge \eta) = (i_X\omega) \wedge \eta + (-1)^k\omega \wedge i_X\eta.$
4. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k\omega \wedge d\eta.$
5. $i_X \circ i_X = 0.$
6. $d \circ d = 0.$
7. $F * \circ d = d \circ F^*.$

Proof. The proof is similar to the proofs we saw for the vector spaces. □

Definition 13.19. A differential form ω is called **closed** if $d\omega = 0$. A differential form $\omega \in \Omega^k(M)$ is called **exact** if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$. Since $d \circ d = 0$, we get that

Every exact form is closed.

13.6 Lie Derivatives of Differential Forms

We derived formulas for computing Lie derivatives of smooth tensor fields in § 12.5, which apply equally well to differential forms. However, in the case of differential forms, the exterior derivative yields a much more powerful formula for computing Lie derivatives, which also has significant theoretical consequences.

So the concepts which we want to understand are how the Lie derivative behaves with:wedge products, interior products and the exterior derivative. We start with the wedge product and recall that the wedge product $\omega \wedge \eta$ is the skew-symmetrization of the tensor product, Moreover, Proposition 12.19 shows that $\mathcal{L}_V(A \otimes B) = \mathcal{L}_V A \otimes B + A \otimes \mathcal{L}_V B$. Thus, we get the following

Proposition 13.20. *Suppose M is a smooth manifold, $V \in \mathfrak{X}(M)$, and $\omega, \eta \in \Omega^*(M)$. Then*

$$\mathcal{L}_V(\omega \wedge \eta) = (\mathcal{L}_V\omega) \wedge \eta + \omega \wedge (\mathcal{L}_V\eta).$$

The next theorem is one of the most important equations about the derivatives of differential forms. It gives a remarkable formula for Lie derivatives of differential forms, which dates back to Élie Cartan, the French mathematician who invented the theory of differential forms.

Theorem 13.21 (Cartan's magic formula). *On a smooth manifold M, for any smooth vector field V and any smooth differential form ω,*

$$\mathcal{L}_V\omega = V \lrcorner (d\omega) + d(V \lrcorner \omega). \tag{13.20}$$

Proof. We prove that (13.20) holds for smooth k-forms by induction on k. We begin with a smooth 0-form f, in which case

$$V \lrcorner (df) + d(V \lrcorner f) = V \lrcorner df = df(V) = Vf = \mathcal{L}_V f,$$

which is (13.20).

Now let $k \geq 1$, and suppose (13.20) has been proved for forms of degree less than k. Let ω be an arbitrary smooth k-form, written in smooth local coordinates as

$$\omega = \sum_I \omega_I \, dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

We want to use the induction hypothesis and the base case so we write $f = x^{i_1}$ and $\beta = \omega_I \, dx^{i_2} \wedge \ldots \wedge dx^{i_k}$ and observe that each term in the expression of ω can be written in the form $df \wedge \beta$, where f is a smooth function and β is a smooth $(k-1)$-form. We know from Corollary 11.18 that $\mathcal{L}_V df = d(\mathcal{L}_V f) = d(Vf)$. Thus,

$$\begin{aligned}\mathcal{L}_V(df \wedge \beta) &= (\mathcal{L}_V df) \wedge \beta + df \wedge (\mathcal{L}_V \beta) \\ &= d(Vf) \wedge \beta + df \wedge (V \lrcorner\, d\beta + d(V \lrcorner\, \beta)).\end{aligned}$$

On the other hand, noting that $V \lrcorner\, df = df(V) = Vf$, we compute

$$\begin{aligned}V \lrcorner\, d(df \wedge \beta) + d(V \lrcorner\, (df \wedge \beta)) &= V \lrcorner\, (-df \wedge d\beta) + d((Vf)\beta - df \wedge (V \lrcorner\, \beta)) \\ &= -(Vf)d\beta + df \wedge (V \lrcorner\, d\beta) + d(Vf) \wedge \beta + (Vf)d\beta + df \wedge d(V \lrcorner\, \beta),\end{aligned}$$

where we used the fact that $V \lrcorner (\omega \wedge \eta) = (V \lrcorner \omega) \wedge \eta + (-1)^k \omega \wedge (V \lrcorner \eta)$. Thus,

$$\mathcal{L}_V(df \wedge \beta) = V \lrcorner\, d(df \wedge \beta) + d(V \lrcorner\, (df \wedge \beta))$$

and (13.20) follows from induction. □

We get the following corollary.

Corollary 13.22 (Lie Derivative Commutes with d). *If V is a smooth vector field and ω is a smooth differential form, then*

$$\mathcal{L}_V(d\omega) = d(\mathcal{L}_V \omega).$$

Proof. This follows from Cartan's formula and the fact that $d \circ d = 0$:

$$\begin{aligned}\mathcal{L}_V d\omega &= V \lrcorner\, d(d\omega) + d(V \lrcorner\, d\omega) = d(V \lrcorner\, d\omega); \\ d\mathcal{L}_V \omega &= d(V \lrcorner\, d\omega) + d(d(V \lrcorner\, \omega)) = d(V \lrcorner\, d\omega).\end{aligned}$$

□

Exercise 13.23. Let M be a smooth manifold and $V \in \mathfrak{X}(M)$.

(a) Show that $\mathcal{L}_V \circ i_V = i_V \circ \mathcal{L}_V$.

(b) Show that $i_V \circ \mathcal{L}_V \omega = i_V(V \lrcorner\, d\omega)$ for any smooth form ω.

Using the invarinat formula (13.19) for $d\omega$ and Cartan's magic formula we get another formula for the Lie derivative of a form.

Proposition 13.24. *If ω is a smooth k-form and $X, Y_1, \ldots, Y_k$ are smooth vector fields on a smooth manifold M, then*

$$(\mathcal{L}_X \omega)(Y_1, \ldots, Y_k) = X(\omega(Y_1, \ldots, Y_k)) - \sum_{i=1}^{k} \omega(Y_1, \ldots, [X, Y_i], \ldots, Y_k). \tag{13.21}$$