13.4 Differential forms on manifolds

After having understood exterior forms on vector spaces and various operations associated to a differential form, we now take $V = T_pM$ where M is a smooth manifold and $p \in M$ and mimic the previous sections. This gives the notion of the space of differential forms on a manifold as well as differential forms on a manifolds.

We denote the subset of T^kT^*M consisting of alternating tensors by Λ^kT^*M:

$$\Lambda^kT^*M = \coprod_{p\in M} \Lambda^k(T_p^*M).$$

As before, for every $p \in M$, $\Lambda_p^kT^*M$ is a $\binom{n}{k}$-dimensional vector space.

Definition 13.12. A **differential k-form** is a tensor field whose value at each point is an alternating tensor. The integer k is called the **degree** of the form and we denote the vector space of smooth k-forms by

$$\Omega^k(M) = \Gamma(\Lambda^kT^*M).$$

The wedge product of two differential forms is defined pointwise: $(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$. Thus, the wedge product of a k-form with an l-form is a $(k + l)$-form. If f is a 0-form and η is a k-form, we interpret the wedge product $f \wedge \eta$ to mean the ordinary product $f\eta$. If we define

$$\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M), \tag{13.11}$$

then the wedge product turns $\Omega^*(M)$ into an associative, anticommutative graded algebra.

In any smooth chart, a k-form ω can be written locally as

$$\omega = \sum_I \omega_I \, dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where the coefficients ω_I are continuous functions defined on the coordinate domain. Note that

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k} \left(\frac{\partial}{\partial x^{j_1}}, \cdots, \frac{\partial}{\partial x^{j_k}} \right) = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_k} \\ \vdots & \ddots & \vdots \\ \delta^{i_k}_{j_1} & \cdots & \delta^{i_k}_{j_k} \end{pmatrix}.$$

and the component functions ω_I of ω are determined by

$$\omega_I = \omega \left(\frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_k}} \right).$$

We just state the expressions for pull-back of differential forms which are just special cases of pullback of tensor fields in § 12.4. Suppose $F\colon M \to N$ is smooth. Then

$$F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta), \tag{13.12}$$

and the local coordinate expressions is

$$F^* \left(\sum_{I-\text{increasing}} \omega_I \, dy^{i_1} \wedge \ldots \wedge dy^{i_k} \right) = \sum_{I-\text{increasing}} (\omega_I \circ F) \, d(y^{i_1} \circ F) \wedge \ldots \wedge d(y^{i_k} \circ F).$$

Let us see an explicit example.

Example 13.13. Define $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ by $F(u,v) = (u, v, u^2 - v^2)$, and let ω be the 2-form $y\,dx \wedge dz + x\,dy \wedge dz$ on $\mathbb{R}^3$. The pullback $F^*\omega$ is computed as follows:

$$\begin{aligned} F^*(y\,dx \wedge dz + x\,dy \wedge dz) &= v\,du \wedge d(u^2 - v^2) + u\,dv \wedge d(u^2 - v^2) \\ &= v\,du \wedge (2u\,du - 2v\,dv) + u\,dv \wedge (2u\,du - 2v\,dv) \\ &= -2v^2\,du \wedge dv + 2u^2\,dv \wedge du = -2(u^2 + v^2)\,du \wedge dv, \end{aligned}$$

where we have used the fact $du \wedge du = dv \wedge dv = 0$ and $du \wedge dv = -dv \wedge du$ by anticommutativity.

Interior multiplication also extends naturally to vector fields and differential forms, simply by letting it act pointwise: if $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$, define a $(k-1)$-form $X \lrcorner\, \omega = i_X\omega$ by

$$(X \lrcorner\, \omega)_p = X_p \lrcorner\, \omega_p.$$

13.5 Exterior Derivatives

In this section we define a natural and probably the most important differential operator on smooth forms, called the **exterior derivative.** It is a generalization of the differential of a function.

For each smooth manifold M, we will show that there is a differential operator $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying $d(d\omega) = 0$ for all ω. We will do this in stages. First we will show below the definition of the operator d on $\mathbb{R}^n$ and then use that to prove the existence of an operator d on any smooth manifold which will satisfy the desired properties from $\mathbb{R}^n$. We will do so by giving an invariant formula for the exterior derivative d.

The definition of d on Euclidean space is as follows.

Definition 13.14. Let $\omega = \sum_J \omega_J\, dx^{j_1} \wedge \cdots \wedge dx^{j_k}$ be a smooth k-form on an open subset $U \subseteq \mathbb{R}^n$, we define its **exterior derivative** $d\omega \in \Omega^{k+1}$ by

$$d\left(\sum_J \omega_J\, dx^{j_1} \wedge \cdots \wedge dx^{j_k}\right) = \sum_J d\omega_J \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}, \tag{13.13}$$

where $d\omega_J$ is the differential of the function ω_J, and hence (13.13) can be written as

$$d\left(\sum_J \omega_J\, dx^{j_1} \wedge \ldots \wedge dx^{j_k}\right) = \sum_J \sum_i \frac{\partial \omega_J}{\partial x^i}\, dx^i \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_k}. \tag{13.14}$$

So for instance, if ω is a 1-form, then (13.14) implies

$$\begin{aligned} d(\omega_j\, dx^j) &= \sum_{i,j} \frac{\partial \omega_j}{\partial x^i}\, dx^i \wedge dx^j \\ &= \sum_{i<j} \frac{\partial \omega_j}{\partial x^i}\, dx^i \wedge dx^j + \sum_{i>j} \frac{\partial \omega_j}{\partial x^i}\, dx^i \wedge dx^j \\ &= \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right) dx^i \wedge dx^j \end{aligned}$$

after we interchange i and j in the second sum and use the fact $dx^j \wedge dx^i = -dx^i \wedge dx^j$. For a smooth 0-form f, (13.14) implies

$$df = \frac{\partial f}{\partial x^i}\, dx^i,$$

which is just the differential of f.

Let us see some properties of the exterior derivative on $\mathbb{R}^n$ before seeing its definition on a manifold.

Proposition 13.15. *The exterior derivative d satisfies the following.*

(a) *d is linear over $\mathbb{R}$.*

(b) *If $\omega \in \Omega^k(U)$ and $\eta \in \Omega^l(U)$, $U \subset \mathbb{R}^n$ open, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta. \tag{13.15}$$

(c) *$d \circ d \equiv 0$. In particular, this tells us that $\operatorname{Im}(d_{k-1}) \subset \ker(d_k)$.*

(d) *d commutes with pullbacks: for $U \underset{\text{open}}{\subset} \mathbb{R}^n$, $V \underset{\text{open}}{\subset} \mathbb{R}^m$, $F \colon U \to V$ a smooth map, and $\omega \in \Omega^k(V)$, we have*

$$F^*(d\omega) = d(F^*\omega). \tag{13.16}$$

Proof. Linearity of d is an immediate consequence of the definition. To prove (b), by linearity it suffices to consider terms of the form $\omega = u\,dx^I \in \Omega^k(U)$ and $\eta = v\,dx^J \in \Omega^l(U)$ for smooth real-valued functions u and v. We compute

$$\begin{aligned}
d(\omega \wedge \eta) &= d((u\,dx^I) \wedge (v\,dx^J)) \\
&= d(uv\,dx^I \wedge dx^J) && \text{(linearity of } \wedge \text{ over } C^\infty(M)) \\
&= (v\,du + u\,dv) \wedge dx^I \wedge dx^J && \text{(product rule)} \\
&= (du \wedge dx^I) \wedge (v\,dx^J) + (-1)^k (u\,dx^I) \wedge (dv \wedge dx^J) && \text{(associativity of } \wedge) \\
&= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta,
\end{aligned}$$

where the $(-1)^k$ comes from the fact that $dv \wedge dx^I = (-1)^k dx^I \wedge dv$ because dv is a 1-form and dx^I is a k-form.

Note that the second order derivatives commute for functions, hence $d \circ d = 0$ for 0-forms, i.e., for $f \in C^\infty(M)$.

We prove (c) first for functions which are 0-forms, which is just a real-valued function. In this case,

$$\begin{aligned}
d(du) = d\left(\frac{\partial u}{\partial x^j}\,dx^j\right) &= \frac{\partial^2 u}{\partial x^i \partial x^j}\,dx^i \wedge dx^j \\
&= \sum_{i<j} \left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{\partial^2 u}{\partial x^j \partial x^i}\right) dx^i \wedge dx^j = 0.
\end{aligned}$$

For the general case, we have

$$\begin{aligned}
d(d\omega) &= d\left(\sum_{J-\text{increasing}} d\omega_J \wedge dx^J\right) \\
&= \sum_{J-\text{increasing}} d(d\omega_J) \wedge dx^J - \sum_{J-\text{increasing}} d\omega_j \wedge d(dx^J) && \text{(using (b))} \\
&= \sum_{J-\text{increasing}} d(d\omega_J) \wedge dx^J + \sum_{J-\text{increasing}} \sum_{i=1}^{k} (-1)^i d\omega_J \wedge dx^{j_1} \wedge \ldots \wedge d(dx^{j_i}) \wedge \ldots \wedge dx^{j_k} = 0,
\end{aligned}$$

where we used the $k = 0$ case for the functions ω_J and each of the coordinate functions x^{j_i}.

Finally, to prove (d), again it suffices to consider $\omega = u\,dx^{i_1} \wedge \ldots \wedge dx^{i_k}$. For such a form, the left-hand side of (13.16) is

$$\begin{aligned}
F^*(d(u\,dx^{i_1} \wedge \ldots \wedge dx^{i_k})) &= F^*(du \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}) \\
&= d(u \circ F) \wedge d(x^{i_1} \circ F) \wedge \ldots \wedge d(x^{i_k} \circ F),
\end{aligned}$$

and the right-hand side is

$$\begin{aligned} d(F^*(u\, dx^{i_1} \wedge \ldots \wedge dx^{i_k})) &= d\left((u \circ F)\, d(x^{i_1} \circ F) \wedge \ldots \wedge d(x^{i_k} \circ F)\right) \\ &= d(u \circ F) \wedge d(x^{i_1} \circ F) \wedge \ldots \wedge d(x^{i_k} \circ F), \end{aligned}$$

so they are equal. □

We now want to mimic the situation in $\mathbb{R}^n$ and define the exterior derivative on manifolds. Obviously, we would like the exterior derivative opearator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ to satisfy the properties in Proposition 13.15. It should also generalize the differential of functions. Unlike the wedge product, the interior product and the pull-back operations that we defined earlier, the exterior derivative is no longer a pointwise operation, but is a local operation (i.e. depends on the "nearby values")

Let $U \underset{\text{open}}{\subset} M$. For $f \in \Omega^0(U) = C^\infty(U)$ we have already seen that $df \in \Omega^1(U)$. So we get a linear map

$$d : \Omega^0(U) \to \Omega^1(U), \quad f \mapsto df.$$

Locally on each coordinate chart we have

$$df = \sum_i (\partial_i f) dx^i,$$

where we use the shorthand notation $\partial_i = \dfrac{\partial}{\partial x^i}$. We also have an "invariant definition" of $df \in \Omega^1(U)$, via

$$df(X) = Xf, \quad \forall X \in \Gamma(TU).$$

Now suppose ω is a k-form on M, so that locally

$$\omega = \sum_I \omega_{i_1,\ldots,i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

We use (13.13) to give the following definition.

Definition 13.16. The exterior derivative of $\omega \in \Omega(U)$ is the $(k+1)$-form $d\omega$ given by the formula

$$\begin{aligned} d\omega &= \sum_I d\omega_{i_1,\ldots,i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \\ &= \sum_{I,i} \partial_i(\omega_{i_1,\ldots,i_k}) dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}. \end{aligned} \tag{13.17}$$

Before we proceed, we need to clarify that $d\omega$ defined above is well-defined. In other words, the $(k+1)$-form $d\omega$ defined above should be independent of the choices of coordinate patches.

We can do it either by checking that (13.17) is unchanged if one use another coordinate chart or we can come up with an equivalent but coordinate-free definition (usually called the invariant formulation). We will take the second approach here which will prove the existence of the operator d on M which when restricted to a coordinate chart (U, x) will give (13.17).

Let's start with small k's to find out the invariant formula of $d\omega$.

- For $k = 0$, i.e. $\omega = f \in C^\infty(U)$, we can regard df as a $C^\infty(U)$-linear map

  $$df : \Gamma(TU) \to C^\infty(U)$$

  such that

  $$df(X) = Xf.$$

- For $k = 1$, i.e. $\omega \in \Omega^1(U)$, $d\omega \in \Omega^2(U)$ and so should have two vector fields as its argument and a function as its output. So
$$d\omega : \Gamma(TU) \times \Gamma(TU) \to C^\infty(U).$$
Let us use the formula (13.17) to see an invariant expression of $d\omega$. We write $\omega = \sum_i \omega_i dx^i$, $X = \sum_k X^k \partial_k$ and $Y = \sum_l Y^l \partial_l$. Then
$$\begin{aligned} d\omega(X,Y) &= \sum_{i,j,k,l} (\partial_j \omega_i) dx^j \wedge dx^i (X^k \partial_k, Y^l \partial_l) \\ &= \sum_{i,j} \big((\partial_j \omega_i) X^j Y^i - (\partial_j \omega_i) X^i Y^j \big) \\ &= \sum_{i,j} \big(X^j \partial_j (\omega_i Y^i) - \omega_i X^j \partial_j (Y^i) - Y^j \partial_j (\omega_i X^i) + \omega_i Y^j \partial_j (X^i) \big) \\ &= X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \qquad \left(\text{using (8.9) } [X,Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \right) \end{aligned}$$
So we arrive at
$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$
- For $k = 2$, i.e. $\omega \in \Omega^2(U)$, by a tedious computation one can prove that as a $C^\infty(U)$-trilinear map
$$d\omega : \Gamma(TU) \times \Gamma(TU) \times \Gamma(TU) \to C^\infty(U),$$
we have
$$\begin{aligned} d\omega(X,Y,Z) =& X(\omega(Y,Z)) - Y(\omega(X,Z)) + Z(\omega(X,Y)) \\ & - \omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X). \end{aligned}$$

So we are led to the following *the invariant formula* for $d\omega$

Theorem 13.17. *For any $\omega \in \Omega^k(U)$, the $(k+1)$-form $d\omega$, viewed as a $C^\infty(U)$-multilinear map*
$$d\omega : \underbrace{\Gamma(TU) \times \ldots \times \Gamma(TU)}_{k+1-times} \to C^\infty(U),$$
is given by the formula
$$d\omega(X_1, \ldots, X_{k+1}) := \sum_i (-1)^{i-1} X_i(\omega(X_1, \ldots, \widehat{X}_i, \ldots, X_{k+1})) \tag{13.18}$$
$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1}). \tag{13.19}$$