Lecture 2

2. Smooth structures

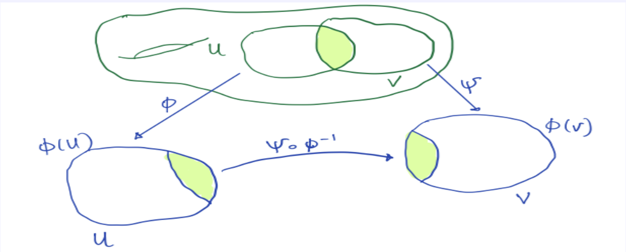
We recall the definition of smooth maps between Euclidean spaces. Recall that if $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open subsets then a map $f : U \to V$ is called **smooth** or **C**$^\infty$ if each of its component functions has continuous partial derivatives of all orders. If, moreover, f is also a bijection and f^{-1} is also smooth then f is said to be a **smooth diffeomorphism**. Since any topological manifold locally looks like an open subset of an Euclidean space, it makes sense to try to define smooth maps on a topological manifold.

We would like to give a definition of when a map $f : M \to \mathbb{R}$ is smooth where M is a topological manifold. For doing this, we would need to put more structure on the manifold. We start with the following

Definition 2.1. Let M be a topological manifold and let (U, ϕ) and (V, ψ) be two charts such that $U \cap V \neq \emptyset$. The map

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \tag{2.1}$$

is called the **transition map from ϕ to ψ.**



We make the following

Definition 2.2. Let $\phi : U \to \tilde{U}$ and $\psi : V \to \tilde{V}$ be two charts on an n-dimensional manifold M. They are called **smoothly compatible** or **C^∞-compatible** if

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \tag{2.2}$$

is a C^∞diffeomorphism.

Having understood the definition of a smoothly compatibe charts , we now define what a atlas and a smooth atlas is.

Definition 2.3. A set of charts $\phi_i : U_i \to \tilde{U}_i$, $i \in I$, of M is called an **atlas** of M, if the domains of the charts cover M, i.e.,

$$\bigcup_{i \in I} U_i = M.$$

An atlas $\mathcal{A}$ is called a **smooth atlas** or a C^∞**-atlas** if any two charts in $\mathcal{A}$ are C^∞-compatible.

For proving that an atlas is a smooth atlas, we would only need to verify that each transition map $\psi \circ \phi^{-1}$ is smooth whenever (U, ϕ) and (V, ψ) are charts in $\mathcal{A}$ as $\psi \circ \phi^{-1}$ is obviously a diffeomorphism because $(\psi \circ \phi^{-1})^{-1} = \phi \circ \psi^{-1}$ and the latter is one of the transition maps which is smooth.

2.1 Examples of smooth atlas

Example 2.4. Let $M = U \subset \mathbb{R}^n$ be open. Then $\mathcal{A} = \{id : U \to U\}$ is a C^∞-atlas.

Example 2.5. For $M = S^n$, we recall the charts from Example 1.6. Consider $\mathcal{A} = \{(x_1 : U_1 \to V_1),\ (x_2 : U_2 \to V_2)\}$

where $U_1 = S^n \setminus \{s\}$ and $U_2 = S^n \setminus \{n\}$. We also recall that

$$\begin{aligned} x_1(y) &= \frac{2}{1+y^0}\hat{y}, \quad \text{where } y = (y^0, \hat{y}) \in \mathbb{R}^{n+1}, \\ x_1^{-1}(z) &= \frac{1}{4+||z||^2}(4-||z||^2, 4z),\ z \in S^n, \\ x_2(y) &= \frac{2}{1-y^0}\hat{y}. \end{aligned}$$

Thus, for any $v \in x_1(U_1 \cap U_2)$, we have

$$\begin{aligned} x_2 \circ x_1^{-1}(v) &= x_2\left(\frac{4-||v||^2}{4+||v||^2}, \frac{4v}{4+||v||^2}\right) \\ &= \frac{2}{1-\frac{4-||v||^2}{4+||v||^2}} \frac{4v}{4+||v||^2} \\ &= \frac{4v}{||v||^2}. \end{aligned}$$

Note that $v \in \mathbb{R}^n \setminus \{0\}$. Thus, $x_2 \circ x_1^{-1}$ is C^∞. Similarly, $x_1 \circ x_2^{-1}$ is also smooth, making x_1 and x_2 C^∞-compatible and $\mathcal{A}$ a C^∞-atlas for S^n.

Example 2.6. Let us recall the real projective space $\mathbb{RP}^n$ and its charts from Example 1.15. One can compute that for say, $i > j$ and $(u^1, ..., u^n) \in (U_i \cap U_j)$, we have

$$\phi_j \circ \phi_i^{-1}(u^1, ..., u^n) = \left(\frac{u^1}{u^j}, \cdots, \frac{u^{j-1}}{u^j}, \frac{u^{j+1}}{u^j}, \cdots, \frac{u^{i-1}}{u^j}, \frac{1}{u^j}, \frac{u^{i+1}}{u^j}, \cdots, \frac{u^n}{u^j}\right),$$

which are just rational functions and hence C^∞. Thus, we get a smooth atlas for $\mathbb{RP}^n$.

We will see more examples later. But for now, recall that we introduced this notion of smooth atlas because we wanted to define when a function $f : M \to \mathbb{R}$ is differentiable. We would like to say that $f : M \to \mathbb{R}$ is smooth if $f \circ \phi^{-1}$ is smooth in the sense of ordinary calculus for each coordinate chart (U, ϕ) in the atlas. There is a small problem with this definition. It can happen that may different atlases give rise to the "same" smooth strcuture on M. To avoid any redundancy, we make the following definition.

Definition 2.7. A C^∞-atlas $\mathcal{A}_{max}$ is called a **maximal atlas** or a **differentiable structure on** M, if every chart that is smoothly compatiblle with all charts in $\mathcal{A}_{max}$ is already contained in $\mathcal{A}_{max}$. In other words,

$$\mathcal{A}_{max} := \{\text{charts } (U, \phi) \text{ in atlas } \mathcal{A} \text{ of } M \mid \phi \text{ is } C^\infty - \text{compatible with all charts in } \mathcal{A}\}. \tag{2.3}$$

Clearly, $\mathcal{A}_{max}$ is also a C^∞-atlas. If ϕ, ψ are two charts of M which are C^∞-compatible with all charts in $\mathcal{A}$ then ϕ and ψ are C^∞-compatible with each other because if $x \in \phi(U \cap V)$ then there exists a chart $\tau : \tilde{U} \to \tilde{\tilde{U}}$ in $\mathcal{A}$ such that $\phi^{-1}(x) \in \tilde{\tilde{U}}$. So, near x, we have

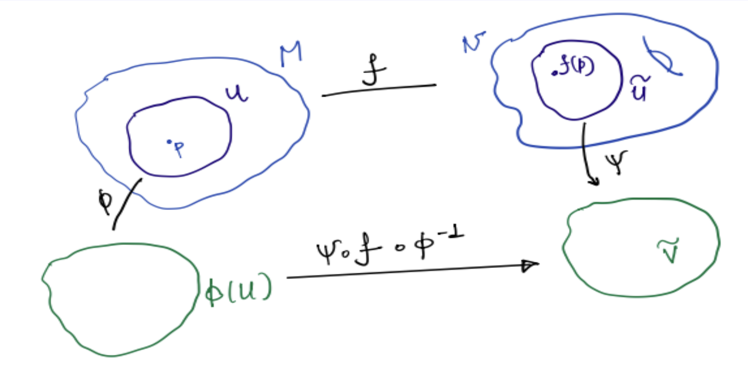
$$\psi \circ \phi^{-1} = \underbrace{(\psi \circ \tau^{-1})}_{C^\infty} \circ \underbrace{(\tau \circ \phi^{-1})}_{C^\infty}.$$

Definition 2.8. Let M^n be a topological manifold. A pair $(M, \mathcal{A}_{max})$, where $\mathcal{A}_{max}$ is a maximal atlas on M, is called an **n-dimensional differentiable manifold.**

With this definition in hand we can finally define the differentiability of functions.

Definition 2.9. Let M and N be differentiable manifolds, $p \in M$ and let $k \in \mathbb{N} \cup \{\infty\}$. A continuous map $f : M \to N$ is called **k-times continuously differentiable at** p or C^k **at** p if $\exists$ a chart $\phi : U \to V \in \mathcal{A}_{max}(M)$ with $p \in U$ and another chart $\psi : \tilde{U} \to \tilde{V} \in \mathcal{A}_{max}(N)$ with $f(p) \in \tilde{U}$ and $f(U) \subset \tilde{U}$ such

$$\psi \circ f \circ \phi^{-1} : \phi(U) \to \tilde{V} \text{ is } C^k. \tag{2.4}$$



Definition 2.10. Let M, N be differentiable manifolds. A homeomorphism $f : M \to N$ is called a **C^k-diffeomorphism** if both f and f^{-1} are C^k. In such a case, M and N are called **C^k-diffeomorphic**.

Before seeing examples of smooth maps, let's see more examples of smooth manifolds.

Remark 2.11. We emphasize that a smooth structure is an additional data which we associate to a topological manifold M. So when we say that M is a "smooth manifold" then it is already understood that there is a smooth maximal atlas prescribed. It can happen that a given topological manifold has many different smooth structures. For example, consider the homeomorphism

$$\psi : \mathbb{R} \to \mathbb{R}, \quad \psi(x) = x^3.$$

The atlas $(\mathbb{R}, \psi)$ is a smooth structure on $\mathbb{R}$ but it is **not C^∞-compatible** with the standard smooth structure $(\mathbb{R}, id)$ on $\mathbb{R}$ because

$$id \circ \psi^{-1}(y) = y^{\frac{1}{3}}$$

is not smooth at the point $\{0\}$. Thus, the two smooth structures are different.

Exercise 2.12. Let M be a nonempty topological manifold of dimension $n \geq 1$. Suppose M has a smooth structure. Show that M has uncountably many distinct smooth structures.

Remark 2.13. In fact, it can happen that a topological manifold has no smooth structures at all. The first example was given by Kervaire [Ker60] on a compact 10-dimensional manifold.

Although, it seems tempting to define a smooth structure by explicitly describing a maximal smooth atlas, it is not always convenient to do so because that can contain many charts and so things might be harder to check explicitly. This can be remedied by the following result.

Proposition 2.14. *Let M be a topological manifold.*

1. *Every smooth atlas $\mathcal{A}$ of M is contained in a unique maximal smooth atlas $\mathcal{A}_{max}$. Such an $\mathcal{A}_{max}$ is also called the* **smooth structure determined by** *$\mathcal{A}$.*
2. *Two smooth atlases of M determine the same smooth structure if and only if their union is a smooth atlas.*

Proof. 1. We have already proved the existence of such $\mathcal{A}_{max}$. To prove uniqueness, note that if $\mathcal{B}$ is any other maximal smooth atlas containing $\mathcal{A}$ then each of its charts is C^∞-compatible with each chart in $\mathcal{A}$ and hence by definition $\mathcal{B} \subset \mathcal{A}_{max}$. Similarly, $\mathcal{A}_{max} \subset \mathcal{B}$ and hence $\mathcal{A}_{max} = \mathcal{B}$.

2. Exercise. ☐

References

[Ker60] Michel A. Kervaire. “A manifold which does not admit any differentiable structure”. English. *Comment. Math. Helv.* 34 (1960), pp. 257–270. DOI: 10.1007/BF02565940. Zbl: 0145.20304 (cit. on p. 11).