We now define the analogue of the symmetrization operator for obtaining alternating tensors.

We define the projection $\operatorname{Alt}\colon T^k(V^*) \to \Lambda^k(V^*)$, called **alternation** or **skew-symmetrization**, as:

$$\operatorname{Alt}\alpha = \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)({}^{\sigma}\alpha),$$

where S_k is the symmetric group on k elements. More explicitly, this means

$$(\operatorname{Alt}\alpha)(v_1,\dots,v_k) = \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\alpha(v_{\sigma(1)},\dots,v_{\sigma(k)}).$$

Thus, for instance, if β is a 2-tensor, then

$$(\operatorname{Alt}\beta)(v,w) = \frac{1}{2}(\beta(v,w) - \beta(w,v)).$$

For a 3-tensor γ,

$$(\operatorname{Alt}\gamma)(v_1,v_2,v_3) = \frac{1}{6}\big(\gamma(v_1,v_2,v_3)+\gamma(v_2,v_3,v_1)+\gamma(v_3,v_1,v_2)-\gamma(v_2,v_1,v_3)-\gamma(v_1,v_3,v_2)-\gamma(v_3,v_2,v_1)\big).$$

We now try to produce an explicit basis of the space $\Lambda^k(V*)$ by introducing the concept of elementary exterior forms.

Notation. For $k\in\mathbb{Z}$, an ordered k-tuple $I=(i_1,\dots,i_k)$ of positive integers is called a **multi-index of length** k. If I is such a multi-index and $\sigma\in S_k$, we write I_σ for the multi-index $I_\sigma=(i_{\sigma(1)},\dots,i_{\sigma(k)})$. Note that $I_{\sigma\tau}=(I_\sigma)_\tau$ for $\sigma,\tau\in S_k$.

Let V be an n-dimensional vector space, and suppose $(\varepsilon^1,\dots,\varepsilon^n)$ is any basis for V^*. We now define a collection of k-exterior forms on V that generalize the determinant function on $\mathbb{R}^n$. For each multi-index $I=(i_1,\dots,i_k)$ of length k such that $1\le i_1,\dots,i_k\le n$, define a covariant k-tensor $\varepsilon^I=\varepsilon^{i_1\dots i_k}$ by

$$\varepsilon^I(v_1,\dots,v_k) = \det\begin{pmatrix}\varepsilon^{i_1}(v_1) & \cdots & \varepsilon^{i_1}(v_k)\\ \vdots & \ddots & \vdots\\ \varepsilon^{i_k}(v_1) & \cdots & \varepsilon^{i_k}(v_k)\end{pmatrix} = \det\begin{pmatrix}v_1^{i_1} & \cdots & v_k^{i_1}\\ \vdots & \ddots & \vdots\\ v_1^{i_k} & \cdots & v_k^{i_k}\end{pmatrix}. \tag{13.3}$$

Since the determinant changes sign whenever two columns are interchanged, it is clear that ε^I is an alternating k-tensor. We call ε^I an **elementary alternating tensor**.

Example 14.5. In terms of the standard dual basis (e^1,e^2,e^3) for $(\mathbb{R}^3)^*$, we have

$$e^{13}(v,w) = v^1w^3 - w^1v^3.$$

Proposition 13.4 (Properties of Elementary k-exterior forms)**.** Let (E_i) be a basis for V, let (ε^i) be the dual basis for V^*, and let ε^I be as defined above.

1. If I has a repeated index, then $\varepsilon^I=0$.
2. If $J=I_\sigma$ for some $\sigma\in S_k$, then $\varepsilon^I=(\operatorname{sgn}\sigma)\varepsilon^J$.
3. The result of evaluating ε^I on a sequence of basis vectors is

$$\varepsilon^I(E_{j_1},\dots,E_{j_k}) = \det\begin{pmatrix}\delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_k}\\ \vdots & \ddots & \vdots\\ \delta^{i_k}_{j_1} & \cdots & \delta^{i_k}_{j_k}\end{pmatrix}.$$

Proof. If I has a repeated index, then for any vectors $v_1, \dots, v_k$, the determinant in (13.3) has two identical rows and thus is equal to zero, which proves (a). On the other hand, if J is obtained from I by interchanging two indices, then the corresponding determinants have opposite signs; this implies (b). Finally, (c) follows immediately from the definition of ε^I. □

We can now use the elementary k-exterior forms to write a basis for $\Lambda^k(V^*)$. A multi-index $I = (i_1, \dots, i_k)$ is said to be **increasing** if $i_1 < \cdots < i_k$.

Proposition 13.5 (A basis for $\Lambda^k(V^*)$)**.** Let V be an n-dimensional vector space. If (ε^i) is any basis for V^*, then for each positive integer $k \le n$, the collection of k-exterior forms

$$\mathcal{E} = \{\varepsilon^I : I \textit{ is an increasing multi-index of length } k\}$$

is a basis for $\Lambda^k(V^*)$. Therefore,

$$\dim \Lambda^k(V^*) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If $k > n$, then $\dim \Lambda^k(V^*) = 0$.

Proof. Note that $\Lambda^k(V^*)$ is the trivial vector space when $k > n$ follows immediately from Proposition 13.3 (b), since every k-tuple of vectors is linearly dependent in that case. For the case $k \le n$, we need to show that the set $\mathcal{E}$ spans $\Lambda^k(V^*)$ and is linearly independent. Let (E_i) be the basis for V dual to (ε^i).

To show that $\mathcal{E}$ spans $\Lambda^k(V^*)$, let $\alpha \in \Lambda^k(V^*)$ be arbitrary. For each multi-index $I = (i_1, \dots, i_k)$ (not necessarily increasing), define a real number α_I by

$$\alpha_I = \alpha(E_{i_1}, \dots, E_{i_k}).$$

Since α is alternating hence $\alpha_I = 0$ if I contains a repeated index, and $\alpha_J = (\operatorname{sgn} \sigma)\alpha_I$ if $J = I_\sigma$ for $\sigma \in S_k$. Thus, for any multi-index J, we have,

$$\sum_{I-\text{increasing}} \alpha_I \varepsilon^I(E_{j_1}, \dots, E_{j_k}) = \alpha_J = \alpha(E_{j_1}, \dots, E_{j_k}).$$

Thus $\sum_{I-\text{increasing}} \alpha_I \varepsilon^I = \alpha$, so $\mathcal{E}$ spans $\Lambda^k(V^*)$. The proof of linear independence is an exercise. □

13.2 The Wedge Product

We now see an operation which allows us to combine two or more exterior forms to again give an exterior form. We still work on a real finite dimensional vector space V. As we see below, the operation of wedge product is just the alternation of tensor product with appropriately chosen normalization constant.

Definition 13.6. Given $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, we define their **wedge product** or **exterior product** to be the following $(k+l)$-exterior form:

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!} \operatorname{Alt}(\omega \otimes \eta). \tag{13.4}$$

One of the reasons we choose this normalization constant in (13.4) is because of the following relation for elementary alternating tensors. One can check that for any multi-indices $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$,

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ},$$

where $IJ = (i_1, \dots, i_k, j_1, \dots, j_l)$ is obtained by concatenating I and J.

We now see some important properties of the wedge product.

Proposition 13.7. Suppose $\omega, \omega', \eta, \eta'$, and ξ are exterior forms on a finite-dimensional vector space V.

1. ***Bilinearity:*** For $a, a' \in \mathbb{R}$,

$$(a\omega + a'\omega') \wedge \eta = a(\omega \wedge \eta) + a'(\omega' \wedge \eta),$$

$$\eta \wedge (a\omega + a'\omega') = a(\eta \wedge \omega) + a'(\eta \wedge \omega').$$

2. ***Associativity:***

$$\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi.$$

3. ***Anticommutativity:*** For $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$,

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega. \tag{13.5}$$

4. If (ε^i) is any basis for V^* and $I = (i_1, \ldots, i_k)$ is any multi-index, then

$$\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} = \varepsilon^I. \tag{13.6}$$

5. For any covectors $\omega^1, \ldots, \omega^k$ and vectors $v_1, \ldots, v_k$,

$$\omega^1 \wedge \cdots \wedge \omega^k(v_1, \ldots, v_k) = \det(\omega^j(v_i)). \tag{13.7}$$

Proof. Since the wedge product is the alternation of tensor product and the tensor product is a bilinear operation, we see bilinearity of the wedge product. We prove 2. and 3. on elementary alternating tensors from which the general case will follows. Note that

$$(\varepsilon^I \wedge \varepsilon^J) \wedge \varepsilon^K = \varepsilon^{IJ} \wedge \varepsilon^K = \varepsilon^{IJK} = \varepsilon^I \wedge \varepsilon^{JK} = \varepsilon^I \wedge (\varepsilon^J \wedge \varepsilon^K),$$

and hence the associativity of wedge product follows. For part 3., we again observe that from the property of being an exterior form, we get

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ} = (\operatorname{sgn} \sigma)\varepsilon^{JI} = (\operatorname{sgn} \sigma)\varepsilon^J \wedge \varepsilon^I,$$

where σ is the permutation that sends IJ to JI. Now the main point is that once you write both I and J as product of 2-cycles, moving J to front and I to back requires moving all the indices of J to front and hence $\operatorname{sgn} \tau = (-1)^{kl}$, because τ. This proves anti-commutativity for alternating tensors and then the general case follows from bilinearity.

Part (4) is an immediate consequence of the fact that $\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}$. Part 5. is left as an exercise. □

Notation. We will write $\varepsilon^I = \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}$.

Definition 13.8. For any n-dimensional vector space V, we define a vector space $\Lambda(V^*)$, called the ***exterior algebra*** (or ***Grassmann algebra***) *of* V. by

$$\Lambda(V^*) = \bigoplus_{k=0}^{n} \Lambda^k(V^*).$$

Note that $\dim \Lambda(V^*) = \binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{n-1} + \binom{n}{n} = 2^n$.
The wedge product is an operation on $\Lambda(V^*)$ and it makes $\Lambda(V^*)$ into an associative algebra but not commutative.

Remark 13.9. An algebra A is said to be **graded** if it has a direct sum decomposition

$$A = \bigoplus_{k \in \mathbb{Z}} A^k$$

such that the product satisfies

$$(A^k)(A^l) \subseteq A^{k+l}$$

for each k and l. A graded algebra is ***anticommutative*** if the product satisfies

$$ab = (-1)^{kl} ba$$

for $a \in A^k$, $b \in A^l$. Thus, $\Lambda(V^*)$ is an anticommutative graded algebra.

We say that a k-form ω is **decomposable** is it can be written as a wedge product of k 1-forms, $\omega = \omega_1 \wedge \cdots \wedge \omega_k$ with ω_i beign 1-forms. Not every k-form is decomposable but every k-form can be written a linear combination of decomposable forms.

13.3 Interior Product

We now briefly discuss the notion of interior product.

Definition 13.10. Let V be a finite-dimensional vector space. For each $v \in V$, we define a linear map

$$i_v \colon \Lambda^k(V^*) \to \Lambda^{k-1}(V^*),$$

called **interior product by** v, as

$$i_v\omega(w_1, \dots, w_{k-1}) = \omega(v, w_1, \dots, w_{k-1}). \tag{13.8}$$

That is, $i_v\omega$ is obtained from ω by inserting v into the first slot. We will also use the notation

$$v \lrcorner \omega = i_v\omega.$$

and read it as “v hook ω."

Let us see some elementary properties of the interior product.

Proposition 13.11. Let V be a finite-dimensional vector space and $v \in V$.

1. $i_v \circ i_v = 0$.
2. *If* $\omega \in \Lambda^k(V^*)$ *and* $\eta \in \Lambda^l(V^*)$*,*

$$v \lrcorner (\omega \wedge \eta) = (v \lrcorner \omega) \wedge \eta + (-1)^k \omega \wedge (v \lrcorner \eta). \tag{13.9}$$

Proof. If ω is a function or a covector then taking interior product twice gives 0. On exterior forms of rank k for $k \geq 2$, notice that $i_v \circ i_v$ means we insert the same vector in the first two slots and since ω is alternating, we get part 1.

We now prove part 2. and note that it suffices to consider the case in which both ω and η are decomposable, since every exterior form can be written as a linear combination of decomposable ones.

Let $v_1 = v \in V$ and pick an arbitrary $(k-1)$-tuple of vectors $(v_2, \dots, v_k)$. Let $\omega^1, \dots, \omega^k$ be 1-forms. We first prove that

$$(\omega^1 \wedge \cdots \wedge \omega^k)(v_1, \dots, v_k) = \sum_{i=1}^{k} (-1)^{i-1} \omega^i(v_1)(\omega^1 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^k)(v_2, \dots, v_k), \tag{13.10}$$

where the hat indicates that ω^i is omitted. To see why (13.10) is true, notice that the LHS is the determinant of the matrix $[A]_{ij}$ whose (i, j)-entry is $\omega^i(v_j)$. To analyze the right-hand side, let v^i_j denote the $(k-1)\times(k-1)$ submatrix of $[A]$ obtained by deleting the ith row and jth column. Then the right-hand side of (13.10) is

$$\sum_{i=1}^{k}(-1)^{i-1}\omega^i(v_1)\det v^i_1.$$

This is just the expansion of $\det[A]$ by minors along the first column, and therefore is equal to $\det[A]$. Thus, (13.10) holds. Thus, we get that

$$v\lrcorner(\omega^1\wedge\cdots\wedge\omega^k)=\sum_{i=1}^{k}(-1)^{i-1}\omega^i(v)\,\omega^1\wedge\cdots\wedge\widehat{\omega^i}\wedge\cdots\wedge\omega^k,$$

which proves part 2 for decomposable forms and hence for all exterior forms. □