12.4 Pullbacks of Tensor Fields

Just like covector fields, covariant tensor fields can be pulled back by a smooth map to yield tensor fields on the domain. This construction works only for covariant tensor fields, which is one reason why we focus most of our attention on the covariant case. The reason is the same as to why the pull-back of a covector field is always defined.

Definition 12.14. Suppose $F\colon M \to N$ is a smooth map. For any point $p \in M$ and any k-tensor $\alpha \in T^k(T^*_{F(p)}N)$, we define a tensor $dF^*_p(\alpha) \in T^k(T^*_pM)$, called the **pointwise pullback of** α **by** F **at** p, by

$$dF^*_p(\alpha)(v_1, \ldots, v_k) = \alpha(dF_p(v_1), \ldots, dF_p(v_k)), \qquad v_1, \ldots, v_k \in T_pM. \tag{12.4}$$

If A is a covariant k-tensor field on N, we define a k-tensor field F^*A on M, called the **pullback of** A **by** F, by

$$(F^*A)_p = dF^*_p(A_{F(p)}),$$

which acts on vectors $v_1, \ldots, v_k \in T_pM$ by

$$(F^*A)_p(v_1, \ldots, v_k) = A_{F(p)}(dF_p(v_1), \ldots, dF_p(v_k)). \tag{12.5}$$

In precisely, the same way as we proved the properties about pullback of vector fields and covector fields, we get the following proposition which collects together some basic properties of pullback of tensor fields.

Proposition 12.15. Suppose $F\colon M \to N$ and $G\colon N \to P$ are smooth maps, A and B are covariant tensor fields on N, C a tensor field on P and $f \in C^\infty(N)$.

(a) $F^*(fA) = (f \circ F)F^*A$.

(b) $F^*(A \otimes B) = F^*A \otimes F^*B$.

(c) $F^*(A + B) = F^*A + F^*B$.

(d) F^*B is a (continuous) tensor field, and is smooth if B is smooth.

(e) $(G \circ F)^*C = F^*(G^*C)$.

(f) $(\mathrm{Id}_N)^*B = B$.

This immediately leads to the following corollary which provides local coordinates description of the pullback of a tensor field and should be compared with equation (11.10) for covector fields.

Corollary 12.16. *Let $F\colon M \to N$ be smooth, and let B be a covariant k-tensor field on N. If $p \in M$ and (y^i) are smooth coordinates for N on a neighborhood of $F(p)$, then F^*B has the following expression in a neighborhood of p:*

$$F^*\left(B_{i_1 \ldots i_k}\, dy^{i_1} \otimes \cdots \otimes dy^{i_k}\right) = (B_{i_1 \ldots i_k} \circ F)\, d(y^{i_1} \circ F) \otimes \cdots \otimes d(y^{i_k} \circ F). \tag{12.6}$$

In general, there is neither a pushforward nor a pullback operation for mixed tensor fields. However, in the special case of a diffeomorphism, tensor fields of any variance can be pushed forward and pulled back. We just state the result and important properties below.

Proposition 12.17. *Let $F : M \to N$ be a diffeomorphism between smooth manifold, Then*

$$F_* : \Gamma(T^{(k,l)}TM) \to \Gamma(T^{(k,l)}TN), \quad F^* : \Gamma(T^{(k,l)}TN) \to \Gamma(T^{(k,l)}TM),$$

generalize the usual pushforwards and pullbacks of vector fields and covector fields respectively such that

1. $F_* = (F^*)^{-1}$.
2. $F^*(A \otimes B) = F^*(A) \otimes F^*(B)$.
3. $(F \circ G)_* = F_* \circ G_*$ *and* $(F \circ G)^* = G^* \circ F^*$.
4. *For any covariant k-tensor A on N,* $F^*(A(X_1, \ldots, X_k)) = F^*A(F_*^{-1}(X_1), \ldots, F_*^{-1}(X_k))$.

12.5 Lie Derivatives of Tensor Fields

Once we know what pullback of tensor fields are, we can mimic the idea about taking Lie derivatives of covector fields in the direction of a vector field and extend it to define the Lie derivative of an arbitrary tensor field in the direction of a vector field.

Suppose M is a smooth manifold, V is a smooth vector field on M, and θ is its flow. For any $p \in M$, if t is sufficiently close to zero, then θ_t is a diffeomorphism from a neighborhood of p to a neighbourhood of $\theta_t(p)$, so $d(\theta_t)_p^*$ pulls back tensors at $\theta_t(p)$ to ones at p by the formula

$$d(\theta_t)_p^*(A_{\theta_t(p)})(v_1, \ldots, v_k) = A_{\theta_t(p)}(d(\theta_t)_p(v_1), \ldots, d(\theta_t)_p(v_k)).$$

where $d(\theta_t)_p^*(A_{\theta_t(p)})$ is just the value of the pullback tensor field $\theta_t^* A$ at p.

Definition 12.18. Given a smooth covariant tensor field A on M, we define the **Lie derivative of A with respect to V**, denoted by $\mathcal{L}_V A$, by

$$(\mathcal{L}_V A)_p = \frac{d}{dt}\bigg|_{t=0} (\theta_t^* A)_p = \lim_{t \to 0} \frac{d(\theta_t)_p^*(A_{\theta_t(p)}) - A_p}{t}, \tag{12.7}$$

provided the derivative exists. The quantity $\mathcal{L}_V A$ is a smooth tensor field on M and is a tensor of the same type as A.

Proposition 12.19. *Let M be a smooth manifold and let $V \in \mathfrak{X}(M)$. Suppose $f \in C^\infty(M)$, and A, B are smooth covariant tensor fields on M. Then*

(a) $\mathcal{L}_V(fA) = (\mathcal{L}_V f)A + f\mathcal{L}_V A$.

(b) $\mathcal{L}_V(A \otimes B) = (\mathcal{L}_V A) \otimes B + A \otimes \mathcal{L}_V B$.

(c) If $X_1, \ldots, X_k$ are smooth vector fields and A is a smooth k-tensor field,

$$\mathcal{L}_V(A(X_1, \ldots, X_k)) = (\mathcal{L}_V A)(X_1, \ldots, X_k) + A(\mathcal{L}_V X_1, \ldots, X_k) + \cdots + A(X_1, \ldots, \mathcal{L}_V X_k). \tag{12.8}$$

(d) *We also have*

$$\begin{aligned}(\mathcal{L}_V A)(X_1, \ldots, X_k) = V(A(X_1, \ldots, X_k)) - A([V, X_1], X_2, \ldots, X_k) - \cdots \\ \cdots - A(X_1, \ldots, X_{k-1}, [V, X_k]).\end{aligned} \tag{12.9}$$

Proof. Erercise, which can be solved in the same way as that for vector fields. □

Let us see an example of how we can compute the Lie derivative of a $(0,2)$-tensor in local coordinates.

Example 12.20. Suppose A is an arbitrary smooth covariant 2-tensor field, and V is a smooth vector field. We compute the Lie derivative $\mathcal{L}_V A$ in smooth local coordinates (x^i). First, we observe that $\mathcal{L}_V dx^i = d(\mathcal{L}_V x^i) = d(Vx^i) = dV^i$. Therefore,

$$\begin{aligned}\mathcal{L}_V A &= \mathcal{L}_V(A_{ij}dx^i \otimes dx^j) \\ &= \mathcal{L}_V(A_{ij})dx^i \otimes dx^j + A_{ij}(\mathcal{L}_V dx^i) \otimes dx^j + A_{ij}dx^i \otimes (\mathcal{L}_V dx^j) \\ &= VA_{ij}\, dx^i \otimes dx^j + A_{ij}\, dV^i \otimes dx^j + A_{ij}\, dx^i \otimes dV^j \\ &= \left(VA_{ij} + A_{kj}\frac{\partial V^k}{\partial x^i} + A_{ik}\frac{\partial V^k}{\partial x^j} \right) dx^i \otimes dx^j.\end{aligned}$$

Thus, once we know the expression of the vector field V in local coordinates and the expression of the tensor A in local coordinates, we can explicitly write down the expression for $\mathcal{L}_V A$ in local coordinates in terms of the derivatives of component functions of V and A. Similar formulas hold for higher rank tensors.

Recall that the Lie derivative of a vector field W with respect to V is zero if and only if W is invariant under the flow of V which was the content of Proposition 10.7. We can say a similar thing for tensors. If A is a smooth tensor field on M and θ is a flow on M, we say that A **is invariant under** θ if for each t, the map θ_t pulls A back to itself wherever it is defined, that is,

$$d(\theta_t)^*_p(A_{\theta_t(p)}) = A_p \tag{12.11}$$

for all (t,p) in the domain of θ. If θ is a global flow, this is equivalent to $\theta_t^* A = A$ for all $t \in \mathbb{R}$.

The following result is an analog of Proposition 10.5 which shows how the Lie derivative can be used to compute time derivatives at times other than $t = 0$.

Proposition 12.21. *Suppose M is a smooth manifold $V \in \mathfrak{X}(M)$. Let θ be the flow of V. For any smooth covariant tensor field A and any (t_0, p) in the domain of θ,*

$$\left.\frac{d}{dt}\right|_{t=t_0} (\theta_t^* A)_p = (\theta_{t_0}^*(\mathcal{L}_V A))_p. \tag{12.10}$$

Proof. Use the definition of the pullback of tensors to notice that (12.10) is equivalent to proving that have to prove

$$\left.\frac{d}{dt}\right|_{t=t_0} d(\theta_t)^*_p(A_{\theta_t(p)}) = d(\theta_{t_0})^*_p((\mathcal{L}_V A)_{\theta_{t_0}(p)}).$$

We mimic the proof of Proposition 10.5 and do a change of variables $t = s + t_0$ to get

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=t_0} d(\theta_t)^*_p(A_{\theta_t(p)}) &= \left.\frac{d}{ds}\right|_{s=0} d(\theta_{s+t_0})^*_p(A_{\theta_{s+t_0}(p)}) \\ &= \left.\frac{d}{ds}\right|_{s=0} d(\theta_{t_0})^*_p d(\theta_s)^*_{\theta_{t_0}(p)}(A_{\theta_s(\theta_{t_0}(p))}) \\ &= d(\theta_{t_0})^*_p \left.\frac{d}{ds}\right|_{s=0} d(\theta_s)^*_{\theta_{t_0}(p)}(A_{\theta_s(\theta_{t_0}(p))}) \\ &= d(\theta_{t_0})^*_p((\mathcal{L}_V A)_{\theta_{t_0}(p)}).\end{aligned}$$

□

Thus, we immediately get the analog of Proposition 10.7 for tensors.

Theorem 12.22. *Let M be a smooth manifold and let $V \in \mathfrak{X}(M)$. A smooth covariant tensor field A is invariant under the flow of V if and only if $\mathcal{L}_V A = 0$.* □

13. Differential forms

We learned about tensors in the previous chapter. Among all tensors, there are specail types of tensors called symmetric and skew-symmetric tensors. In this chapter, we will learn about the latter. But first, we define both symmetric and skew-symmetric tensors.

Definition 13.1. A rank k-covariant tensor α on a finite-dimensional real vector space V is called **symmetric** if

$$\alpha(v_1, ..., v_i, ..., v_j, ..., v_k) = \alpha(v_1, ..., v_j, ..., v_i, ..., v_k) \tag{13.1}$$

for all $1 \leq i < j \leq k$.

In fact, (13.1) implies that the value of α is unchanged when the vectors $v_1, ..., v_k$ are rearranged in any order. In terms of the component functions, this means that $\alpha_{i_1...i_k}$ are unchanged by any permutation of indices.

The set of symmetric covariant k-tensors $\Sigma^k(V^*)$ is a linear subspace of the space $T^k(V^*)$ of all covariant k-tensors on V. Given any tensor α we can make it into a symmetric tensor as follows. Let S_k denote the symmetric group on k elements. Given a k-tensor α and a permutation $\sigma \in S_k$, we define a new k-tensor ${}^\sigma\alpha$ by

$${}^\sigma\alpha(v_1, \ldots, v_k) = \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Note that ${}^\tau({}^\sigma\alpha) = {}^{\tau\sigma}\alpha$, where $\tau\sigma$ represents the composition of τ and σ, that is, $\tau\sigma(i) = \tau(\sigma(i))$.

We define a projection $\operatorname{Sym}\colon T^k(V^*) \to \Sigma^k(V^*)$ called **symmetrization** by

$$\operatorname{Sym}\alpha = \frac{1}{k!}\sum_{\sigma\in S_k} {}^\sigma\alpha = \frac{1}{k!}\sum_{\sigma\in S_k} \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Clearly $\operatorname{Sym}\alpha$ is a symmetric tensor.

If α and β are symmetric tensors on V, then $\alpha \otimes \beta$ is not symmetric in general. For example, consider α, β as covectors. Individually, both are symmetric, but

$$\alpha \otimes \beta(v_1, v_2) = \alpha(v_1)\beta(v_2) \neq \alpha \otimes \beta(v_2, v_1).$$

But we can define the symmetrization of $\alpha \otimes \beta$ as above. If $\alpha \in \Sigma^k(V^*)$ and $\beta \in \Sigma^l(V^*)$, we define their ***symmetric product*** to be the $(k+l)$-tensor $\alpha\beta$ by

$$\alpha\beta = \operatorname{Sym}(\alpha \otimes \beta).$$

which acts on vectors $v_1, \ldots, v_{k+l}$ is given by

$$\alpha\beta(v_1, \ldots, v_{k+l}) = \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}} \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}).$$

Thus, for example, if α and β are covectors, then

$$\alpha\beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha).$$

The same analysis on each tangent space gives the notion of symmetric tensors on manifolds. We already saw an example before: Riemannian metric on a manifold. We will learn more about this later.

13.1 Alternating Tensors

Let us first discuss various notions related to alternating tensors on a vector space and then we will just translate all the knowledge to manifolds. We have the following definition.

Definition 13.2. A covariant k-tensor α on V is said to be **alternating** or **antisymmetric** if it changes sign whenever two of its arguments are interchanged, that is, for all vectors $v_1, \dots, v_k \in V$ and every pair of distinct indices i, j we have satisfies

$$\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k). \tag{13.2}$$

Alternating covariant k-tensors are also called **exterior forms**.

Recall that for any permutation $\sigma \in S_k$, the **sign** of σ, denoted by $\operatorname{sgn} \sigma$, is equal to $+1$ if σ is even and -1 if σ is odd. Thus, (13.2) implies that if α is alternating then for any vectors $v_1, \dots, v_k$ and any permutation $\sigma \in S_k$,

$$\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = (\operatorname{sgn} \sigma)\alpha(v_1, \dots, v_k),$$

and the components $\alpha_{i_1 \dots i_k}$ of α change sign whenever two indices are interchanged.
Notation. The vector space of all alternating k-tensors on V is denoted by $\Lambda^k(V^*)$. Clearly all 0-tensors and 1-tensors are alternating.

Proposition 13.3. Let α be a covariant k-tensor on a finite-dimensional vector space V. The following are equivalent:

(a) α *is alternating.*

(b) $\alpha(v_1, \dots, v_k) = 0$ *whenever the k-tuple* $(v_1, \dots, v_k)$ *is linearly dependent.*

(c) α *gives the value zero whenever two of its arguments are equal:*

$$\alpha(v_1, \dots, w, \dots, w, \dots, v_k) = 0.$$

Proof. The implications (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (c) are immediate. We complete the proof by showing that (c) implies both (a) and (b).

Assume that α satisfies (c). For any vectors $v_1, \dots, v_k$, the hypothesis implies

$$\begin{aligned} 0 &= \alpha(v_1, \dots, v_i + v_j, \dots, v_i + v_j, \dots, v_k) \\ &= \alpha(v_1, \dots, v_i, \dots, v_i, \dots, v_k) + \alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \\ &\quad + \alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k) + \alpha(v_1, \dots, v_j, \dots, v_j, \dots, v_k) \\ &= \alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) + \alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k). \end{aligned}$$

Thus α is alternating. On the other hand, if $(v_1, \dots, v_k)$ is a linearly dependent k-tuple, then one of the v_i's can be written as a linear combination of the others. For simplicity, let us assume that

$$v_k = \sum_{j=1}^{k-1} a^j v_j.$$

Then multilinearity of α implies

$$\alpha(v_1, \dots, v_k) = \sum_{j=1}^{k-1} a^j \alpha(v_1, \dots, v_{k-1}, v_j).$$

In each of these terms, α has two identical arguments, so every term is zero. □