A corollary of Proposition 12.2 is the following statement on basis of the space of tensors.

**Corollary 12.7.** *Let $V$ be an $n$-dimensional real vector space with a basis $(E_i)$ and let $(\varepsilon^j)$ is the dual basis for $V^*$. Then the following sets constitute bases for the tensor spaces over $V$:*

$$\left\{\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} : 1 \leq i_1, \ldots, i_k \leq n\right\} \qquad \text{for } T^k(V^*);$$

$$\left\{E_{i_1} \otimes \cdots \otimes E_{i_k} : 1 \leq i_1, \ldots, i_k \leq n\right\} \qquad \text{for } T^k(V);$$

$$\left\{E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l} : 1 \leq i_1, \ldots, i_k, j_1, \ldots, j_l \leq n\right\} \qquad \text{for } T^{(k,l)}(V).$$

Therefore,

$$\dim T^k(V^*) = \dim T^k(V) = n^k \quad \text{and} \quad \dim T^{(k,l)}(V) = n^{k+l}.$$

Thus, given a basis of $V$ and the corresponding dual basis, we can explicitly write the expression of any element $A \in T^k(V^*)$. We have

$$A = A_{i_1 \ldots i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k},$$

where the $n^k$ coefficients $A_{i_1 \ldots i_k}$ are determined by

$$A_{i_1 \ldots i_k} = A(E_{i_1}, \ldots, E_{i_k}).$$

## 12.3 Tensor Fields on Manifolds

Let $M$ be a smooth manifold. We mimic our descriptions of tangent bundle and cotangent bundle to first define the **bundle of covariant $k$-tensors on $M$** by

$$T^k T^* M = \coprod_{p \in M} T^k(T_p^* M).$$

Analogously, we define the **bundle of contravariant $k$-tensors** by

$$T^k T M = \coprod_{p \in M} T^k(T_p M),$$

and the **bundle of mixed tensors of type $(k,l)$** by

$$T^{(k,l)} T M = \coprod_{p \in M} T^{(k,l)}(T_p M).$$

Immediately, we see, for instance that

$$T^{(0,0)} T M = T^0 T^* M = T^0 T M = M \times \mathbb{R},$$

$$T^{(0,1)} T M = T^1 T^* M = T^* M,$$

$$T^{(1,0)} T M = T^1 T M = T M,$$

$$T^{(0,k)} T M = T^k T^* M,$$

$$T^{(k,0)} T M = T^k T M.$$

Just like the case with the tangent and cotangent bundle, one can prove that the above space, also called **tensor bundle over $M$** are smooth manifolds with a smooth structure inherited from $M$ and they also come equipped with a projection map $\pi : T^{(k,l)}(M) \to M$. Thus, we can also mimic the description of vector fields and covector fields to give the following definition.

**Definition 12.8.** A $(k,l)$-tensor field $\alpha$ on a smooth manifold $M$ is a continuous map

$$\alpha : M \to T^{(k,l)}(M)$$

such that

$$\pi \circ \alpha = id_M.$$

The tensor field $\alpha$ is smooth if it is a smooth map between smooth manifolds.

We see that contravariant 1-tensor fields are the same as vector fields, and covariant 1-tensor fields are covector fields, a 0-tensor field is the same as a continuous real-valued function.

**Notation:** We will denote the space of all $(k,l)$-tensor fields by $\Gamma(T^{(k,l)}TM)$. These are infinite-dimensional vector spaces over $\mathbb{R}$, and are modules over$C^\infty(M)$. In any smooth local coordinates $(x^i)$, elements of these bundles can be written (using the summation convention) as

$$A = \begin{cases} A_{i_1\dots i_k}\, dx^{i_1}\otimes\cdots\otimes dx^{i_k}, & A \in \Gamma(T^kT^*M); \\ A^{i_1\dots i_k}\,\dfrac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\dfrac{\partial}{\partial x^{i_k}}, & A\in\Gamma(T^kTM); \\ A^{i_1\dots i_k}_{j_1\dots j_l}\,\dfrac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\dfrac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_l}, & A\in\Gamma(T^{(k,l)}TM). \end{cases}$$

The functions $A_{i_1\dots i_k}$, $A^{i_1\dots i_k}$, or $A^{i_1\dots i_k}_{j_1\dots j_l}$ are called the **component functions of** $A$ in the chosen coordinates. For brevity, we will denote the space of all smooth covariant $k$-tensor fields by

$$\mathcal{T}^k(M) = \Gamma(T^kT^*M).$$

The next proposition is analogous to Proposition 8.8 which gave the smoothness criteria for vector fields or Proposition 11.5 for covector fields. We only state this for covariant tensors but the result is similar for other types of tensors.

**Proposition 12.9.** *Let $M$ be a smooth manifold and let $A\colon M\to T^kT^*M$ be a tensor field. The following are equivalent.*

*(1)* $A$ is smooth.

*(2)* In every smooth coordinate chart, the component functions of $A$ are smooth.

*(3)* If $X_1,\dots,X_k\in\mathfrak{X}(M)$, then the function $A(X_1,\dots,X_k)\colon M\to\mathbb{R}$, defined by

$$A(X_1,\dots,X_k)(p) = A_p(X_1|_p,\dots,X_k|_p),$$

is smooth.

*(4)* Whenever $X_1,\dots,X_k$ are smooth vector fields defined on some open subset $U\subseteq M$, the function $A(X_1,\dots,X_k)$ is smooth on $U$.

Thus, we again relegated the smoothness for a tensor field to either looking at its component functions or cooking up a function out of $A$ by feeding $k$ vector fields to it and then looking at the smoothness of the resultant function. Similar to the case of vector fields, we have the following

**Proposition 12.10.** *Suppose $M$ is a smooth manifold, $A\in\mathcal{T}^k(M)$, $B\in\mathcal{T}^l(M)$, and $f\in C^\infty(M)$. Then $fA$ and $A\otimes B$ are also smooth tensor fields, whose components in any smooth local coordinate chart are*

$$(fA)_{i_1\dots i_k} = fA_{i_1\dots i_k},$$

$$(A\otimes B)_{i_1\dots i_{k+l}} = A_{i_1\dots i_k}B_{i_{k+1}\dots i_{k+l}}.$$

Now a question arises: if we see a map which takes $k$ vector fields as its arguments and gives us a smooth function, how can we guarantee that it is indeed a smooth covariant tensor field of rank $k$? First of all, we notice that any rank $k$-covariant tensor $A$ induces a map

$$\underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k \text{ copies}} \longrightarrow C^\infty(M).$$

which is multilinear over $\mathbb{R}$. But, in fact, more is true: it is *multilinear over* $C^\infty(M)$, which means that for $f, f' \in C^\infty(M)$ and $X_i, X_i' \in \mathfrak{X}(M)$, we have

$$A(X_1, \ldots, fX_i + f'X_i', \ldots, X_k) = fA(X_1, \ldots, X_i, \ldots, X_k) + f'A(X_1, \ldots, X_i', \ldots, X_k).$$

This property turns out to be characteristic of smooth tensor fields, as the next lemma shows, which we state without proof.

**Lemma 12.11** (Tensor Characterization Lemma)**.** *A map*

$$\mathcal{A}\colon \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k \text{ copies}} \longrightarrow C^\infty(M), \tag{12.3}$$

*is induced by a smooth covariant $k$-tensor field as above if and only if it is multilinear over $C^\infty(M)$.*

A **symmetric tensor field** on a manifold is a covariant tensor field whose value at each point is a symmetric tensor, i.e., the value of the tensor doesn't change if we change the position of the vectors in its argument. The symmetric product of two or more tensor fields is defined pointwise, just like the tensor product. Thus, for example, if $A$ and $B$ are smooth covector fields, their symmetric product is the smooth 2-tensor field $AB$, which is given by

$$AB = \frac{1}{2}(A \otimes B + B \otimes A).$$

Let us mention two more examples of tensors.

**Example 12.12 (Riemannian metrics).** Let $M$ be a manifold. A Riemannian metric $g$ on $M$ associates to every point $p \in M$ an inner product $g_p$ on $T_pM$ and thus we get a bilinear map

$$g_p : T_p \times T_pM \to \mathbb{R}$$

which is also symmetric and positive-definite. A Riemannian metric is called smooth if the function $M \to \mathbb{R} : p \mapsto g(X(p), Y(p))$ is smooth for any pair of vector fields $X, Y \in \mathfrak{X}(M)$. Thus, $g$ is an example of a $(0, 2)$-tensor which is symmetric.

**Example 12.13 (Almost complex structure).** Suppose $V$ is a $2n$-dimensional real vector space. We can make it into an $n$-dimensional complex vector space if we have a map $J : V \to V$ such that $J^2 = -id$ and then defining $(a + ib)v = av + bJV$. Such a $J$ is called a **complex structure** on $V$.

An **almost complex structure** on $M$ is a map $J : \Gamma(TM) \to \Gamma(TM)$ such that it a complex structure on $T_pM$ for all $p \in M$. Thus an almost complex structure is an example of a $(1, 1)$-tensor field.