11.4 Pullbacks of Covector Fields

As we have seen, a smooth map yields a linear map on tangent vectors called the differential. Dualizing this leads to a linear map on covectors going in the opposite direction.

Let $F\colon M \to N$ be a smooth map between smooth manifolds and let $p \in M$ be arbitrary. The differential $dF_p\colon T_pM \to T_{F(p)}N$ yields a dual linear map

$$dF_p^*\colon T_{F(p)}^*N \to T_p^*M,$$

called the **(pointwise) pullback by** F **at** p, or the **cotangent map of** F. Unraveling the definitions, we see that dF_p^* is characterized by

$$dF_p^*(\omega)(v) = \omega\big(dF_p(v)\big), \quad \text{for } \omega \in T_{F(p)}^*N,\ v \in T_pM.$$

Here comes the important part. Recall that when we discussed vector fields, we made a point of noting that pushforwards of vector fields under smooth maps are defined only in the special cases of diffeomorphisms or Lie group homomorphisms (recall the notion of being F-related). The surprising thing about covectors is that covector fields *always* pull back to covector fields.

Definition 11.11. Given a smooth map $F\colon M \to N$ and a covector field ω on N, define a covector field $F^*\omega$ on M, called the ***pullback of*** ω ***by*** F, by

$$(F^*\omega)_p = dF_p^*\left(\omega_{F(p)}\right). \tag{11.7}$$

It acts on a vector $v \in T_pM$ by

$$(F^*\omega)_p(v) = \omega_{F(p)}\big(dF_p(v)\big).$$

In contrast to the vector field case, there is no ambiguity here about what point to pull back from: the value of $F^*\omega$ at p is the pullback of ω at $F(p)$.

Let us prove two important properties of the pullback.

Proposition 11.12. *Let $F\colon M \to N$ be a smooth map between smooth manifolds. Suppose u is a continuous real-valued function on N, and ω is a covector field on N. Then*

$$F^*(u\omega) = (u \circ F)F^*\omega. \tag{11.8}$$

If in addition u is smooth, then

$$F^*du = d(u \circ F). \tag{11.9}$$

Proof. To prove (11.8) we compute

$$\begin{aligned}
\big(F^*(u\omega)\big)_p &= dF_p^*\big((u\omega)_{F(p)}\big) \\
&= dF_p^*\big(u(F(p))\omega_{F(p)}\big) \\
&= u\big(F(p)\big)dF_p^*\big(\omega_{F(p)}\big) && \text{(by linearity of } dF_p^*) \\
&= u\big(F(p)\big)(F^*\omega)_p && \text{(by definition)} \\
&= \big((u \circ F)F^*\omega\big)_p.
\end{aligned}$$

For (11.9), we let $v \in T_pM$ be arbitrary, and compute

$$\begin{aligned}
(F^*du)_p(v) &= \big(dF_p^*\big(du_{F(p)}\big)\big)(v) && \text{(by (11.7))} \\
&= du_{F(p)}\big(dF_p(v)\big) && \text{(by definition of } dF_p^*) \\
&= dF_p(v)u && \text{(by definition of } du) \\
&= v(u \circ F) && \text{(by definition of } dF_p) \\
&= d(u \circ F)_p(v) && \text{(by definition of } d(u \circ F)).
\end{aligned}$$

□

Proposition 11.13. *Suppose $F\colon M \to N$ is a smooth map between smooth manifolds and let ω be a covector field on N. Then $F^*\omega$ is a covector field on M. If ω is smooth, then so is $F^*\omega$.*

Proof. Let $p \in M$ be arbitrary, and choose smooth coordinates (y^j) for N in a neighborhood V of $F(p)$. Let $U = F^{-1}(V)$, which is a neighborhood of p. Writing ω in coordinates as $\omega = \omega_j\, dy^j$ for continuous functions ω_j on V and using Proposition 11.12 twice, we have the following computation in U:

$$F^*\omega = F^*\left(\omega_j\, dy^j\right) = (\omega_j \circ F) F^*\, dy^j = (\omega_j \circ F) d\left(y^j \circ F\right). \tag{11.10}$$

This expression is continuous, and is smooth if ω is smooth. □

Example 11.14. Let $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the map given by

$$(u, v) = F(x, y, z) = \left(x^2 y,\ y \sin z\right),$$

and let $\omega \in \mathfrak{X}^*\left(\mathbb{R}^2\right)$ be the covector field

$$\omega = u\, dv + v\, du.$$

According to (11.10), the pullback $F^*\omega$ is given by

$$\begin{aligned} F^*\omega &= (u \circ F) d(v \circ F) + (v \circ F) d(u \circ F) \\ &= \left(x^2 y\right)\, d(y \sin z) + (y \sin z) d\left(x^2 y\right) \\ &= x^2 y (\sin z\, dy + y \cos z\, dz) + y \sin z \left(2xy\, dx + x^2\, dy\right) \\ &= 2xy^2 \sin z\, dx + 2x^2 y \sin z\, dy + x^2 y^2 \cos z\, dz. \end{aligned}$$

Suppose M is a smooth manifold, $S \subseteq M$ is an immersed submanifold and $\iota\colon S \hookrightarrow M$ is the inclusion map. If ω is any smooth covector field on M, the pullback by ι yields a smooth covector field $\iota^*\omega$ on S. To see what this means, let $v \in T_p S$ be arbitrary, and compute

$$(\iota^*\omega)_p(v) = \omega_p\big(d\iota_p(v)\big) = \omega_p(v),$$

since $d\iota_p\colon T_p S \to T_p M$ is just the inclusion map, under our usual identification of $T_p S$ with a subspace of $T_p M$. Thus, $\iota^*\omega$ is just the restriction of ω to vectors tangent to S. For this reason, $\iota^*\omega$ is often called the ***restriction of*** ω ***to*** S.

11.5 Lie Derivatives of 1-forms

The Lie derivative operation can be extended to covector fields. Suppose M is a smooth manifold, V is a smooth vector field on M, and θ is its flow. For any $p \in M$, if t is sufficiently close to zero, then θ_t is a diffeomorphism from a neighborhood of p to a neighbourhood of $\theta_t(p)$, so $d(\theta_t)_p^*$ pulls back covectors at $\theta_t(p)$ to ones at p by the formula

$$d(\theta_t)_p^*(\omega_{\theta_t(p)})(v) = \omega_{\theta_t(p)}(d(\theta_t)_p(v_1)).$$

Note that $d(\theta_t)_p^*(\omega_{\theta_t(p)})$ is just the value of the pullback covector field $\theta_t^*\omega$ at p.

Definition 11.15. Given a smooth covector field ω on M, we define the ***Lie derivative of*** ω ***with respect to*** V, denoted by $\mathcal{L}_V\omega$, by

$$(\mathcal{L}_V\omega)_p = \frac{d}{dt}\bigg|_{t=0} (\theta_t^*\omega)_p = \lim_{t\to 0} \frac{d(\theta_t)_p^*(\omega_{\theta_t(p)}) - \omega_p}{t}, \tag{11.11}$$

provided the derivative exists.

Just as for the case of vector fields we have

Proposition 11.16. *With M, V, and ω as above, the derivative in (11.11) exists for every $p \in M$ and defines $\mathcal{L}_V\omega$ as a smooth covector field on M.*

Lemma 11.17. *Let M be a smooth manifold and let $V \in \mathfrak{X}(M)$. Suppose f is a smooth real-valued function on M, and ω is a smooth covector field on M. Then*

(a) $\mathcal{L}_V(f\omega) = (\mathcal{L}_V f)\omega + f\mathcal{L}_V\omega$.

(b) If X is a smooth vector field then,

$$\mathcal{L}_V(\omega(X)) = (\mathcal{L}_V\omega)(X) + \omega(\mathcal{L}_V X) \tag{11.12}$$

One consequence of this proposition is the following formula expressing the Lie derivative of any smooth covector field in terms of Lie brackets and ordinary directional derivatives of functions, which allows us to compute Lie derivatives without first determining the flow.

Corollary 11.18. *We have the following.*

1. *If V is a smooth vector field and ω is a smooth covector field, then for any smooth vector fields X*

$$(\mathcal{L}_V\omega)(X) = V(\omega(X)) - \omega([V, X]). \tag{11.13}$$

2. *If $f \in C^\infty(M)$, then $\mathcal{L}_V(df) = d(\mathcal{L}_V f)$.*

Proof. We just prove (2). Using (11.13), for any $X \in \mathfrak{X}(M)$ we compute

$$\begin{aligned}(\mathcal{L}_V df)(X) &= V(df(X)) - df([V, X]) = VXf - [V, X]f \\ &= VXf - (VXf - XVf) = XVf \\ &= d(Vf)(X) = d(\mathcal{L}_V f)(X).\end{aligned}$$

□

12. Tensors

After having studied the concept of vector fields and covector fields in detail and understanding the techniques to study them, we now start with the study of tensors which provide a uniform way to combine the study of tangent and cotangent bundles. We begin with tensors on a vector space, which are multilinear generalizations of covectors; a covector is the special case of a tensor of rank one.

Many geometric structures which we will see in the study of differential geometry are tensors: for instance, Riemannian metrics, almost complex structures, symplectic structures, differential forms etc.

12.1 Multilinear Algebra

Recall that covectors are real-valued linear functions on a vector space. A tensor is a generalization of this in the sense that tensors are real-valued *multilinear* functions of one or more variables.

Suppose $V_1, \ldots, V_k$, and W are vector spaces. A map

$$F\colon V_1 \times \cdots \times V_k \to W$$

is said to be *multilinear* if it is linear as a function of each variable for each i,

$$F(v_1, \ldots, av_i + bv_i', \ldots, v_k) = aF(v_1, \ldots, v_i, \ldots, v_k) + bF(v_1, \ldots, v_i', \ldots, v_k).$$

In this way, a multilinear function of one variable is just a linear function, and a multilinear function of two variables is generally called *bilinear*. Let us write $\mathrm{L}(V_1, \ldots, V_k; W)$ for the set of all multilinear maps from $V_1 \times \cdots \times V_k$ to W. It is a vector space under the usual operations of pointwise addition and scalar multiplication:

$$(F + F')(v_1, \ldots, v_k) = F(v_1, \ldots, v_k) + F'(v_1, \ldots, v_k),$$
$$(aF)(v_1, \ldots, v_k) = a(F(v_1, \ldots, v_k)).$$

For example, dot product in $\mathbb{R}^n$, the cross product in $\mathbb{R}^3$, the determinant are all examples of multilinear maps. Let's see a different but rather important example.

Example 12.1 (Tensor product of covectors). Suppose V is a vector space, and $\omega, \eta \in V^*$. Define a function $\omega \otimes \eta\colon V \times V \to \mathbb{R}$ by

$$\omega \otimes \eta(v_1, v_2) = \omega(v_1)\eta(v_2),$$

where the product on the right is just ordinary multiplication of real numbers. The linearity of ω and η guarantees that $\omega \otimes \eta$ is a bilinear function of v_1 and v_2. If (e^1, e^2) denotes the standard dual basis for $(\mathbb{R}^2)^*$, then $e^1 \otimes e^2\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is the bilinear function

$$e^1 \otimes e^2((w, x), (y, z)) = wz.$$

We can generalize the last example to any arbitrary real-valued multilinear functions as follows: let $V_1, \ldots, V_k, W_1, \ldots, W_l$ be real vector spaces, and suppose $F \in \mathrm{L}(V_1, \ldots, V_k; \mathbb{R})$ and $G \in \mathrm{L}(W_1, \ldots, W_l; \mathbb{R})$. Define a function

$$F \otimes G\colon V_1 \times \cdots \times V_k \times W_1 \times \cdots \times W_l \to \mathbb{R}$$

by

$$F \otimes G(v_1, \ldots, v_k, w_1, \ldots, w_l) = F(v_1, \ldots, v_k)G(w_1, \ldots, w_l). \tag{12.1}$$

It follows from the multilinearity of F and G that $F \otimes G(v_1, \ldots, v_k, w_1, \ldots, w_l)$ depends linearly on each argument v_i or w_j separately, so $F \otimes G$ is an element of $\mathrm{L}(V_1, \ldots, V_k, W_1, \ldots, W_l; \mathbb{R})$, called the **tensor product of F and G**. One can show that

$$(F \otimes G) \otimes H = F \otimes (G \otimes H).$$

Thus, we can write tensor products of three or more multilinear functions unambiguously without parentheses. If $F_1, \ldots, F_l$ are multilinear functions depending on $k_1, \ldots, k_l$ variables, respectively, their tensor product $F_1 \otimes \cdots \otimes F_l$ is a multilinear function of $k = k_1 + \cdots + k_l$ variables, whose action on k vectors is given by inserting the first k_1 vectors into F_1, the next k_2 vectors into F_2, and so forth, and multiplying the results together.

If $\omega^j \in V_j^*$ for $j = 1, \dots, k$, then $\omega^1 \otimes \cdots \otimes \omega^k \in \mathrm{L}(V_1, \dots, V_k; \mathbb{R})$ is the multilinear function given by

$$\omega^1 \otimes \cdots \otimes \omega^k(v_1, \dots, v_k) = \omega^1(v_1) \cdots \omega^k(v_k). \tag{12.1}$$

The next proposition shows that a basis for any space of multilinear functions can be formed by taking all possible tensor products of basis covectors.

Proposition 12.2 (A Basis for the Space of Multilinear Functions). *Let $V_1, \dots, V_k$ be real vector spaces of dimensions $n_1, \dots, n_k$, respectively. For each $j \in \{1, \dots, k\}$, let $(E_1^{(j)}, \dots, E_{n_j}^{(j)})$ be a basis for V_j, and let $(\varepsilon_{(j)}^1, \dots, \varepsilon_{(j)}^{n_j})$ be the corresponding dual basis for V_j^*. Then the set*

$$\mathcal{B} = \left\{ \varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k} : 1 \le i_1 \le n_1, \dots, 1 \le i_k \le n_k \right\}$$

is a basis for $\mathrm{L}(V_1, \dots, V_k; \mathbb{R})$, which therefore has dimension equal to $n_1 \cdots n_k$.

Proof. We need to show that $\mathcal{B}$ is linearly independent and spans $\mathrm{L}(V_1, \dots, V_k; \mathbb{R})$. Suppose $F \in \mathrm{L}(V_1, \dots, V_k; \mathbb{R})$ is arbitrary. For each ordered k-tuple $(i_1, \dots, i_k)$ of integers with $1 \le i_j \le n_j$, define a number $F_{i_1 \dots i_k}$ by

$$F_{i_1 \dots i_k} = F\left(E_{i_1}^{(1)}, \dots, E_{i_k}^{(k)}\right). \tag{12.2}$$

If we show that

$$F = F_{i_1 \dots i_k} \varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k}$$

the it will follow that $\mathcal{B}$ spans $\mathrm{L}(V_1, \dots, V_k; \mathbb{R})$. For any k-tuple of vectors $(v_1, \dots, v_k) \in V_1 \times \cdots \times V_k$, write $v_1 = v_1^{i_1} E_{i_1}^{(1)}, \dots, v_k = v_k^{i_k} E_{i_k}^{(k)}$, and compute

$$\begin{aligned} F_{i_1 \dots i_k} \varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k}(v_1, \dots, v_k) &= F_{i_1 \dots i_k} \varepsilon_{(1)}^{i_1}(v_1) \cdots \varepsilon_{(k)}^{i_k}(v_k) \\ &= F_{i_1 \dots i_k} v_1^{i_1} \cdots v_k^{i_k}, \end{aligned}$$

while $F(v_1, \dots, v_k)$ is equal to the same thing by multilinearity. This proves the claim.

To show that $\mathcal{B}$ is linearly independent, suppose some linear combination equals zero:

$$F_{i_1 \dots i_k} \varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k} = 0.$$

Apply this to any ordered k-tuple of basis vectors, $(E_{j_1}^{(1)}, \dots, E_{j_k}^{(k)})$. By the same computation as above, this implies that each coefficient $F_{j_1 \dots j_k}$ is zero. Thus, the only linear combination of elements of $\mathcal{B}$ that sums to zero is the trivial one. □

This proof shows, by the way, that the components $F_{i_1 \dots i_k}$ of a multilinear function F in terms of the basis elements in $\mathcal{B}$ are given by (12.2). Thus, F is completely determined by its action on all possible sequences of basis vectors.

12.2 Covariant and Contravariant Tensors

Definition 12.3. Let V be a finite-dimensional real vector space. If k is a positive integer, a *covariant k-tensor on V* is an element of the k-fold tensor product $V^* \otimes \cdots \otimes V^*$, which can also be thoght of as a real-valued multilinear function of k elements of V:

$$\alpha\colon \underbrace{V \times \cdots \times V}_{k \text{ copies}} \longrightarrow \mathbb{R}.$$

The number k is called the *rank of α*.

Thus, a covriant tensor eats k-vectors and spits out a real number. A 0-tensor is, by convention, just a real number. We denote the vector space of all covariant k-tensors on V by the notation

$$T^k(V^*) = \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ copies}}.$$

For instance, a **covector** is a covariant tensor of rank 1. An inner product is a covariant tensor of rank 2.

Definition 12.4. For any finite-dimensional real vector space V, we define the space of *contravariant tensors on V of rank k* to be the vector space

$$T^k(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ copies}}.$$

Elements of $T^k(V)$ can also be thought of as

$$T^k(V) \cong \left\{ \text{multilinear functions } \alpha\colon \underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \to \mathbb{R} \right\}.$$

Combining the previous two definitions we have the following

Definition 12.5. For any nonnegative integers k, l, we define the space of *mixed tensors on V of type (k, l)* as

$$T^{(k,l)}(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ copies}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l \text{ copies}}.$$

Thus, in our notation, for instance,

$$T^{(0,0)}(V) = T^0(V^*) = T^0(V) = \mathbb{R},$$

$$T^{(0,1)}(V) = T^1(V^*) = V^*,$$
$$T^{(1,0)}(V) = T^1(V) = V,$$
$$T^{(0,k)}(V) = T^k(V^*),$$
$$T^{(k,0)}(V) = T^k(V).$$

Remark 12.6. The notation $T^{(k,l)}(V)$ is not universal. Another common notation is $T^k_l(V)$. Another notation reverse the roles of k and l so one must be careful in reading another source.