11. Cotangent bundle and 1-forms

Recall that if V is a finite-dimensional $\mathbb{R}$-vector space then the space of all linear functionals, i.e., linear maps $\omega : V \to \mathbb{R}$ is denoted by V^* and is called the dual space to V. Elements of V^* are called **covectors** on V. In fact, we immediately get a basis of V^* once we choose a basis of V as given in the next proposition.

Proposition 11.1. *Let V be a finite-dimensional vector space. Given any basis $(e_1, \ldots, e_n)$ for V, let $\varepsilon^1, \ldots, \varepsilon^n \in V^*$ be the covectors defined by*

$$\varepsilon^i(e_j) = \delta^i_j,$$

*where δ^i_j is the Kronecker delta symbol. Then $(\varepsilon^1, \ldots, \varepsilon^n)$ is a basis for V^*, called the **dual basis** to (e_j) and hence $\dim V^* = \dim V$.*

So, for instance, for the standard basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$, the dual basis is denoted by $\left(e^1, \ldots, e^n\right)$, and is called the ***standard dual basis***. Notice that we are using upper indices for denoting the covectors. These basis covectors are the linear functionals on $\mathbb{R}^n$ given by

$$e^i(v) = e^i\left(v^1, \ldots, v^n\right) = v^i.$$

In matrix notation, a linear map from $\mathbb{R}^n$ to $\mathbb{R}$ is represented by a $1 \times n$ matrix, that is, it is a ***row matrix***. The basis covectors can therefore also be thought of as the linear functionals represented by the row matrices

$$e^1 = (1\ 0\ \ldots\ 0), \quad e^2 = (0\ 1\ 0\ \ldots\ 0), \quad \ldots, \quad e^n = (0\ \ldots\ 0\ 1).$$

In general, if (e_j) is a basis for V and (ε^i) is its dual basis, then for any vector $v = v^j e_j \in V$, we have

$$\varepsilon^i(v) = v^j\, \varepsilon^i(e_j) = v^j\, \delta^i_j = v^i.$$

We can express an arbitrary covector $\omega \in V^*$ in terms of the dual basis as

$$\omega = \omega_i\, \varepsilon^i, \tag{11.1}$$

where the components are determined by $\omega_i = \omega(e_i)$ and similarly, the action of ω on a vector $v = v^j e_j$ is

$$\omega(v) = \omega_i\, v^i. \tag{11.2}$$

Notation: We always write basis covectors with upper indices, and components of a covector with lower indices.
Suppose V and W are vector spaces and $L\colon V \to W$ is a linear map. We define a linear map $L^*\colon W^* \to V^*$, called the ***dual map*** or ***transpose of L***, by

$$(L^*\omega)(v) = \omega(Lv) \quad \text{for } \omega \in W^*,\ v \in V.$$

The dual maps satisfies the following properties.

(1) $(A \circ B)^* = B^* \circ A^*$.

(2) $(\mathrm{Id}_V)^*\colon V^* \to V^*$ *is the identity map of* V^*.

For a vector space V, we also have a ***second dual space*** $V^{**} = (V^*)^*$. For each vector space V there is a natural, basis-independent map $\xi\colon V \to V^{**}$, defined as follows. For each vector $v \in V$, define a linear functional $\xi(v)\colon V^* \to \mathbb{R}$ by

$$\xi(v)(\omega) = \omega(v) \quad \text{for } \omega \in V^*.$$

Proposition 11.2. *For any finite-dimensional vector space V, the map $\xi\colon V \to V^{**}$ is an isomorphism.*

Proof. Because $\dim V = \dim V^{**}$, it suffices to verify that ξ is injective. Suppose $v \in V$ is not zero. Extend v to a basis $(v = e_1, \dots, e_n)$ for V, and let $(\varepsilon^1, \dots, \varepsilon^n)$ denote the dual basis for V^*. Then $\xi(v) \neq 0$ because

$$\xi(v)\left(\varepsilon^1\right) = \varepsilon^1(v) = \varepsilon^1(e_1) = 1.$$

□

The preceding proposition shows that when V is finite-dimensional, we can unambiguously identify V^{**} with V itself, because the map ξ is canonically defined, without reference to any basis. It is important to observe that although V^* is also isomorphic to V (for the simple reason that any two finite-dimensional vector spaces of the same dimension are isomorphic), there is no *canonical* isomorphism $V \cong V^*$.

11.1 Tangent Covectors on Manifolds

Now let M be a smooth manifold. For each $p \in M$, we define the ***cotangent space at*** p, denoted by T_p^*M, to be the dual space to T_pM

$$T_p^*M = (T_pM)^*\,.$$

Elements of T_p^*M are called ***tangent covectors at*** p, or just ***covectors at*** p.

Let (U, x^i) be a coordinate chart. Then the coordinate basis $\left(\frac{\partial}{\partial x^i}\big|_p\right)$ gives rise to a dual basis for T_p^*M, which we denote by $\left(\lambda^i\big|_p\right)$. Any covector $\omega \in T_p^*M$ can thus be written uniquely as $\omega = \omega_i\, \lambda^i\big|_p$, where

$$\omega_i = \omega\left(\frac{\partial}{\partial x^i}\bigg|_p\right).$$

Let us try to understand how the covectors look like under change of coordinate system. Suppose that $(\tilde{x}^j)$ is another set of smooth coordinates whose domain contains p, and let $\left(\tilde{\lambda}^j\big|_p\right)$ denote the basis for T_p^*M dual to $\left(\frac{\partial}{\partial \tilde{x}^j}\big|_p\right)$. We recall that the coordinate vector fields transform as follows:

$$\frac{\partial}{\partial x^i}\bigg|_p = \frac{\partial \tilde{x}^j}{\partial x^i}(p)\, \frac{\partial}{\partial \tilde{x}^j}\bigg|_p. \tag{11.3}$$

Writing ω in both systems as $\omega = \omega_i\, \lambda^i\big|_p = \tilde{\omega}_j\, \tilde{\lambda}^j\big|_p$, we can use (11.3) to compute the components ω_i in terms of $\tilde{\omega}_j$:

$$\omega_i = \omega\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \omega\left(\frac{\partial \tilde{x}^j}{\partial x^i}(p)\, \frac{\partial}{\partial \tilde{x}^j}\bigg|_p\right) = \frac{\partial \tilde{x}^j}{\partial x^i}(p)\tilde{\omega}_j. \tag{11.4}$$

Thus, the components of a vector, i.e., the n-tuples $(v^1, \dots, v^n)$ and $(\tilde{v}^1, \dots, \tilde{v}^n)$ assigned to two different coordinate systems (x^i) and $(\tilde{x}^j)$ were related by the transformation law

$$\tilde{v}^j = \frac{\partial \tilde{x}^j}{\partial x^i}(p)v^i,$$

and a covector transforms, according to the following slightly different rule:

$$\omega_i = \frac{\partial \tilde{x}^j}{\partial x^i}(p)\tilde{\omega}_j. \tag{11.7}$$

11.2 Covector Fields or 1-forms

For any smooth manifold M the disjoint union

$$T^*M = \coprod_{p \in M} T_p^*M$$

is called the ***cotangent bundle of*** M. It has a natural projection map $\pi\colon T^*M \to M$ sending $\omega \in T_p^*M$ to $p \in M$. As above, given any smooth local coordinates (x^i) on an open subset $U \subseteq M$, for each $p \in U$ we denote the basis for T_p^*M dual to $\left(\frac{\partial}{\partial x^i}\big|_p\right)$ by $\left(\lambda^i\big|_p\right)$. This defines n maps $\lambda^1, \ldots, \lambda^n\colon U \to T^*M$, called ***coordinate covector fields***. Just like the case for tangent bundle Theorem 4.8, the next result states that the cotangent bundle is also a $2n$-dimensional manifold with a smooth structure inherited from M.

Theorem 11.3. *Let M be a smooth n-manifold. The cotangent bundle T^*M has a unique topology and smooth structure making it into a smooth $2n$-dimensional manifold.*

Proof. We leave the proof as an exercise. □

As in the case of the tangent bundle, smooth local coordinates for M yield smooth local coordinates for its cotangent bundle. If (x^i) are smooth coordinates on an open subset $U \subseteq M$,then the map from $\pi^{-1}(U)$ to $\mathbb{R}^{2n}$ given by

$$\xi_i \, \lambda^i\big|_p \mapsto \left(x^1(p), \ldots, x^n(p), \xi_1, \ldots, \xi_n\right)$$

is a smooth coordinate chart for T^*M. We call (x^i, ξ_i) the ***natural coordinates for*** T^*M associated with (x^i).

Again, as was the case with vector fields, we have the following definition.

Definition 11.4. A **covector field** or a **(differential) 1-form** is a continuous map $\omega : M \to T^*M$ such that

$$\pi \circ \omega = \mathrm{id}_M.$$

Thus, for every $p \in M$, $\omega_p \in T_p^*M$.

In any smooth local coordinates on an open subset $U \subseteq M$, a covector field ω can be written in terms of the coordinate covector fields (λ^i) as $\omega = \omega_i \, \lambda^i$ for n functions $\omega_i\colon U \to \mathbb{R}$ called the ***component functions of ω***. They are characterized by

$$\omega_i(p) = \omega_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right).$$

If ω is a covector field and X is a vector field on M, then we can form a function $\omega(X)\colon M \to \mathbb{R}$ by

$$\omega(X)(p) = \omega_p(X_p), \quad p \in M.$$

If we write $\omega = \omega_i \, \lambda^i$ and $X = X^j \frac{\partial}{\partial x^j}$ in terms of local coordinates, then $\omega(X)$ has the local coordinate representation $\omega(X) = \omega_i \, X^i$.

Just as in the case of vector fields, there are several ways to check for smoothness of a covector field.

Proposition 11.5. *Let M be a smooth manifold and let $\omega\colon M \to T^*M$ be a covector field. The following are equivalent:*

1. *ω is smooth.*
2. *In every smooth coordinate chart, the component functions of ω are smooth.*
3. *For every smooth vector field $X \in \mathfrak{X}(M)$, the function $\omega(X)$ is smooth on M.*
4. *For every open subset $U \subseteq M$ and every smooth vector field X on U, the function $\omega(X)\colon U \to \mathbb{R}$ is smooth on U.*

We denote the real vector space of all smooth covector fields on M by $\mathfrak{X}^*M$ or $\Omega^1(M)$. For $f \in C^\infty(M)$, the covector field $f\omega$ is defined as

$$(f\omega)_p = f(p)\omega_p. \tag{11.5}$$

11.3 The Differential of a Function

Recall that the gradient of a smooth real-valued function f on $\mathbb{R}^n$ is defined as the vector field whose components are the partial derivatives of f

$$\operatorname{grad} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}.$$

Although the partial derivatives of a smooth function cannot be interpreted in a coordinate-independent way as the components of a vector field, it turns out that they can be interpreted as the components of a covector field. This is the most important application of covector fields.

Definition 11.6. Let f be a smooth real-valued function on a smooth manifold M We define a covector field df, called the ***differential of f***, by

$$df_p(v) = vf \quad \text{for } v \in T_pM.$$

Proposition 11.7. *The differential of a smooth function is a smooth covector field.*

Proof. Clearly at each point $p \in M$, $df_p(v)$ depends linearly on v, so that df_p is indeed a covector at p. To see that df is smooth, we use Proposition 11.5 (3): for any smooth vector field X on M, the function $df(X)$ is smooth because it is equal to Xf. □

Let us compute df in coordinates or in other words, let's compute its coordinate representation. Let (x^i) be smooth coordinates on an open subset $U \subseteq M$, and let (λ^i) be the corresponding coordinate dual basis on U. Write df in coordinates as $df_p = A_i(p)\,\lambda^i\big|_p$ for some functions $A_i\colon U \to \mathbb{R}$; then the definition of df implies

$$A_i(p) = df_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial f}{\partial x^i}(p).$$

This yields the following formula for the coordinate representation of df:

$$df_p = \frac{\partial f}{\partial x^i}(p)\,\lambda^i\bigg|_p. \tag{11.6}$$

Thus, the component functions of df in any smooth coordinate chart are the partial derivatives of f with respect to those coordinates. As a result, the differential of f in coordinates is analogous to the gradient of f in local coordinates but df makes sense irrespective of any coordinate system.

Applying (11.6) to the special case in which f is one of the coordinate functions $x^j\colon U \to \mathbb{R}$, we obtain

$$dx^j\big|_p = \frac{\partial x^j}{\partial x^i}(p)\,\lambda^i\big|_p = \delta^j_i\,\lambda^i\big|_p = \lambda^j\big|_p.$$

In other words, the coordinate covector field λ^j is none other than the differential dx^j. Therefore, the formula (11.6) for df_p can be rewritten as

$$df_p = \frac{\partial f}{\partial x^i}(p)\,dx^i\big|_p,$$

or as an equation between covector *fields* instead of covectors:

$$df = \frac{\partial f}{\partial x^i}\,dx^i. \tag{11.11}$$

From now on, we abandon the notation λ^i for the coordinate coframe, and use dx^i instead.

Example 11.8. If $f(x, y) = x^2 y \cos x$ on $\mathbb{R}^2$, then df is given by the formula

$$df = \frac{\partial \left(x^2 y \cos x\right)}{\partial x}\, dx + \frac{\partial \left(x^2 y \cos x\right)}{\partial y}\, dy$$
$$= \left(2xy \cos x - x^2 y \sin x\right) dx + x^2 \cos x \, dy.$$

Recall that for a smooth real-valued function $f\colon M \to \mathbb{R}$, we now have two different definitions for the differential of f at a point $p \in M$. Previously, we defined df_p as a linear map from T_pM to $T_{f(p)}\mathbb{R}$.Now, we defined df_p as a covector at p, which is to say a linear map from T_pM to $\mathbb{R}$. These are really the same object, once we take into account the canonical identification between $\mathbb{R}$ and $T_{f(p)}\mathbb{R}$; one easy way to see this is to note that both are represented in coordinates by the row matrix whose components are the partial derivatives of f.

Proposition 11.9 (Properties of the Differential)**.** *Let M be a smooth manifold and let $f, g \in C^\infty(M)$.*

(a) If a and b are constants, then $d(af + bg) = a\,df + b\,dg$.

(b) $d(fg) = f\,dg + g\,df$.

(c) $d(f/g) = (g\,df - f\,dg)/g^2$ on the set where $g \neq 0$.

(d) If f is constant, then $df = 0$.

Proposition 11.10 (Functions with Vanishing Differentials)**.** *If f is a smooth real-valued function on a smooth manifold M then $df = 0$ if and only if f is constant on each component of M.*

Proof. It suffices to assume that M is connected and show that $df = 0$ if and only if f is constant. One direction is immediate: if f is constant, then $df = 0$. Conversely, suppose $df = 0$, let $p \in M$, and let $\mathcal{C} = \{q \in M : f(q) = f(p)\}$. If q is any point in $\mathcal{C}$, let U be a smooth coordinate ball centred at q. From (11.11) we see that $\frac{\partial f}{\partial x^i} \equiv 0$ in U for each i, so by elementary calculus f is constant on U. This shows that $\mathcal{C}$ is open, and since it is closed by continuity, it must be all of M. Thus, f is everywhere equal to the constant $f(p)$. □