Just like Lemma 9.8, we have the following result.

Lemma 9.13. *If $\theta\colon \mathcal{D} \to M$ is a smooth flow, then the infinitesimal generator V of θ is a smooth vector field, and each curve $\theta^{(p)}$ is an integral curve of V.*

We make the following definition.

Definition 9.14. A **maximal integral curve** is one that cannot be extended to an integral curve on any larger open interval and a **maximal flow** is a flow that admits no extension to a flow on a larger flow domain.

The next theorem is the main result of about flows of vector fields.

Theorem 9.15 (Fundamental Theorem on Flows). *Let V be a smooth vector field on a smooth manifold M. There is a unique smooth maximal flow $\theta\colon \mathcal{D} \to M$ whose infinitesimal generator is V. This flow has the following properties:*

(a) For each $p \in M$, the curve $\theta^{(p)}: \mathcal{D}^{(p)} \to M$ is the unique maximal integral curve of V starting at p.

(b) If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(\theta(s,p))}$ is the interval $\mathcal{D}^{(p)} - s = \{t - s : t \in \mathcal{D}^{(p)}\}$.

(c) For each $t \in \mathbb{R}$, the set

$$M_t = \{p \in M \mid (t,p) \in \mathcal{D}\}$$

is open in M, and $\theta_t\colon M_t \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}.

(d) For each $(t,p) \in \mathcal{D}$, $(\theta_t)_* V(p) = V(\theta(p))$.

Proof. The proof is a bit long so we proceed in stages. We first prove uniqueness in part (a). We know from Proposition 9.5 that given the smooth vector field V, there exists an integral curve starting at each point $p \in M$ for atleast some short time. Suppose $\gamma, \widetilde{\gamma}\colon J \to M$ are two integral curves of V defined on the same open interval J such that $\gamma(t_0) = \widetilde{\gamma}(t_0)$ for some $t_0 \in J$. We want to prove that in this case when $\gamma, \widetilde{\gamma}$ agree at one point, they must agree on every point of their common domain. Let

$$\mathcal{S} = \{t \in J \mid \gamma(t) = \widetilde{\gamma}(t)\}.$$

Clearly, $\mathcal{S} \neq \emptyset$, because $t_0 \in \mathcal{S}$ by hypothesis, and $\mathcal{S}$ is closed in J by continuity as if (t_n) is a sequence in $\mathcal{S}$ which converges to t, we can use continuity of $\gamma, \widetilde{\gamma}$ to get $\gamma(t_n) \to \gamma(t) = \widetilde{\gamma}(t_n) = \widetilde{\gamma}(t)$, thus proving that $t \in \mathcal{S}$. We now want to prove that $\mathcal{S}$ is also open and then use connectedness of J to prove that $J = \mathcal{S}$. Suppose $t_1 \in \mathcal{S}$. Then in a smooth coordinate neighbourhood around the point $p = \gamma(t_1)$, γ and $\widetilde{\gamma}$ are both solutions to same ODE with the same initial condition $\gamma(t_1) = \widetilde{\gamma}(t_1) = p$. By the uniqueness part of Theorem 9.4, $\gamma \equiv \widetilde{\gamma}$ on an interval containing t_1, which implies that $\mathcal{S}$ is open in J. Since J is connected, $\mathcal{S} = J$, which implies that $\gamma = \widetilde{\gamma}$ on all of J. Thus, any two integral curves that agree at one point agree on their common domain. Thus, if we manage to define a flow θ and then the coorresponding $\theta^{(p)}$ then it must be the unique integral curve starting at p.

We now prove maximality of the integral curve.Let $p \in M$ and let $\mathcal{D}^{(p)}$ be the union of all open intervals $J \subseteq \mathbb{R}$ containing 0 on which an integral curve starting at p is defined. Since we want to prove maximality of the domain, a natural thing would be to take the union and that is exactly what we are doing above. Define $\theta^{(p)}: \mathcal{D}^{(p)} \to M$ by letting $\theta^{(p)}(t) = \gamma(t)$, where γ is any integral curve starting at p and defined on an open interval containing 0 and t. Given any point t, since all such integral curves agree at t as explained above, $\theta^{(p)}$ is well defined,, i.e., it's image is independent of the integral curve chosen and is obviously the unique maximal integral curve starting at p as we have taken the union of all possible intervals.

Now let

$$\mathcal{D} = \{(t,p) \in \mathbb{R} \times M : t \in \mathcal{D}^{(p)}\},$$

and define $\theta\colon \mathcal{D} \to M$ by $\theta(t,p) = \theta^{(p)}(t)$. By definition, for each $p \in M$, $\theta^{(p)}$ is the unique maximal integral curve of V starting at p. To verify that θ is actually a flow, we will need to verify group laws and show that $\mathcal{D}$ is a flow domain. Fix any $p \in M$ and $s \in \mathcal{D}^{(p)}$, and write $q = \theta(s,p) = \theta^{(p)}(s)$. The curve $\gamma\colon \mathcal{D}^{(p)} - s \to M$ defined by $\gamma(t) = \theta^{(p)}(t+s)$ starts at q, and the translation property of integral curves shows that γ is an integral curve of V. By uniqueness of ODE solutions, γ agrees with $\theta^{(q)}$ on their common domain, which is equivalent to the second group law (9.7), and the first group law (9.6) is a by-product of the definition. Thus, θ is actually a flow.

By maximality of $\theta^{(q)}$, the domain of γ cannot be larger than $\mathcal{D}^{(q)}$, which means that $\mathcal{D}^{(p)} - s \subseteq \mathcal{D}^{(q)}$. Since $0 \in \mathcal{D}^{(p)}$, this implies that $-s \in \mathcal{D}^{(q)}$, and the group law implies that $\theta^{(q)}(-s) = \theta^{(p)}(s-s) = \theta^{(p)}(0) = p$. Applying the same argument with $(-s, q)$ in place of (s, p), we find that $\mathcal{D}^{(q)} + s \subseteq \mathcal{D}^{(p)}$, which is the same as $\mathcal{D}^{(q)} \subseteq \mathcal{D}^{(p)} - s$. This proves part (b).

We continue the proof of showing that the flow θ obtained above indeed is a smooth flow. That is, we show that $\mathcal{D}$ is open in $\mathbb{R} \times M$ (so it is a flow domain), and that $\theta \colon \mathcal{D} \to M$ is smooth. Define a subset $W \subseteq \mathcal{D}$ as the set of all $(t, p) \in \mathcal{D}$ such that θ is defined and smooth on a product neighbourhood of (t, p) of the form $J \times U \subseteq \mathcal{D}$, where $J \subseteq \mathbb{R}$ is an open interval containing 0 and $U \subseteq M$ is a neighbourhood of p. Then W is open in $\mathbb{R} \times M$, and the restriction of θ to W is smooth because of the smoothness of each integral curve $\theta^{(p)}$ which follows from the smoothness of the vector field V, so it suffices to show that $W = \mathcal{D}$. Suppose this is not the case. Then there exists some point $(\tau, p_0) \in \mathcal{D} \setminus W$. Without loss of generality, assume $\tau > 0$.

Let $t_0 = \inf\{t \in \mathbb{R}_+ : (t, p_0) \notin W\}$ (note that this is different from what we saw in the lectures because we added $\mathbb{R}_+$ in the definition, you can also take $\sup$ instead of $\inf$ and proceed similarly. The main idea remains unchanged.). By the ODE theorem, θ is defined and smooth in some product neighbourhood of $(0, p_0)$, so $t_0 > 0$. Since $t_0 \le \tau$ and $\mathcal{D}^{(p_0)}$ is an open interval containing 0 and τ, it follows that $t_0 \in \mathcal{D}^{(p_0)}$. Let $q_0 = \theta^{(p_0)}(t_0)$. By the ODE theorem again, there exist $\varepsilon > 0$ and a neighbourhood U_0 of q_0 such that $(-\varepsilon, \varepsilon) \times U_0 \subseteq W$. We will use the group law to show that θ extends smoothly to a neighborhood of (t_0, p_0), which contradicts our choice of t_0. We will do this by using the translation property of integral curves.

Choose some $t_1 < t_0$ such that $t_1 + \varepsilon > t_0$ and $\theta^{(p_0)}(t_1) \in U_0$. We can do this because of the neighbourhood of smoothness being $(-\varepsilon, \varepsilon) \times U_0$ for the point q_0. Since $t_1 < t_0$, we have $(t_1, p_0) \in W$, as t_0 is the infimum, and so there is a product neighborhood $(t_1 - \delta, t_1 + \delta) \times U_1 \subseteq W$. By definition of W, this implies that θ is defined and smooth on $[0, t_1 + \delta) \times U_1$. Because $\theta(t_1, p_0) \in U_0$, we can choose U_1 small enough that θ maps $\{t_1\} \times U_1$ into U_0. Define $\widetilde{\theta} \colon [0, t_1 + \varepsilon) \times U_1 \to M$ by

$$\widetilde{\theta}(t, p) = \begin{cases} \theta_t(p), & p \in U_1,\ 0 \le t < t_1, \\ \theta_{t - t_1} \circ \theta_{t_1}(p), & p \in U_1,\ t_1 - \varepsilon < t < t_1 + \varepsilon. \end{cases}$$

The group law for θ guarantees that these definitions agree where they overlap, and our choices of U_1, t_1, and ε ensure that this defines a smooth map. By the translation property, each map $t \mapsto \widetilde{\theta}(t, p)$ is an integral curve of V, so $\widetilde{\theta}$ is a smooth extension of θ to a neighbourhood of (t_0, p_0), contradicting our choice of t_0. This completes the proof that $W = \mathcal{D}$ and completes the proof of the statement that θ is a smooth flow and $\mathcal{D}$ as defined above is a flow domain.

We now prove (c). The fact that M_t is open is an immediate consequence of the fact that $\mathcal{D}$ is open as each slice at any fixed time t is precisely M_t. We now prove that θ_t indeed maps M_t to M_{-t}. From part (b) we deduce

$$\begin{aligned} p \in M_t \Rightarrow t \in \mathcal{D}^{(p)} &\Rightarrow \mathcal{D}^{(\theta_t(p))} = \mathcal{D}^{(p)} - t \\ &\Rightarrow -t \in \mathcal{D}^{(\theta_t(p))} \Rightarrow \theta_t(p) \in M_{-t}, \end{aligned}$$

which shows that θ_t maps M_t to M_{-t}. Moreover, the group laws then show that $\theta_{-t} \circ \theta_t$ is equal to the identity on M_t. Reversing the roles of t and $-t$ shows that $\theta_t \circ \theta_{-t}$ is the identity on M_{-t}, which proves that θ is a diffeomorphism.

Finally, we prove part (d). Let $\theta(t_0, p) = q$ so we would like to prove that $(\theta_{t_0})_*(V(p)) = V(\theta_{t_0}(p)) = V(q)$. Let $f \in C^\infty(M)$. Then

$$\begin{aligned} ((\theta_{t_0})_*(V(p)))f = V_p(f \circ \theta_{t_0}) &= \left.\frac{d}{dt}\right|_{t=0} f \circ \theta_{t_0} \circ \theta^p(t) \\ &= \left.\frac{d}{dt}\right|_{t=0} f(\theta_{t_0 + t}(p)) && \text{(using the group law)} \\ &= \left.\frac{d}{dt}\right|_{t=0} f(\theta^q(t)) \\ &= \left(\left.\frac{d}{dt}\right|_{t=0} \theta^q(t) \right) f = V(q) f && \text{(as } \theta^q \text{ is an integral curve for } V\text{),} \end{aligned}$$

thus completeing the proof of the theorem. □

Definition 9.16. The flow θ in Theorem 9.15 is called the ***flow generated by V***, or just the ***flow of V***.

Heuristically, the term "infinitesimal generator" comes from fact that in a smooth chart, a good approximation to an integral curve can be obtained by composing many small straight-lines, with the direction and length of each motion determined by the value of the vector field at the point arrived at in the previous step. Intuitively, one can think of a flow as a sequence of infinitely many infinitesimally small linear steps and hence V is "generating" the flow θ.

Proposition 9.17 (Naturality of Flows)**.** *Suppose M and N are smooth manifolds, $F\colon M \to N$ is a smooth map, $X \in \mathfrak{X}(M)$, and $Y \in \mathfrak{X}(N)$. Let θ be the flow of X and η the flow of Y. If X and Y are F-related, then for each $t \in \mathbb{R}$, $F(M_t) \subseteq N_t$ and $\eta_t \circ F = F \circ \theta_t$ on M_t:*

$$\begin{array}{ccc} M_t & \xrightarrow{F} & N_t \\ \theta_t \downarrow & & \downarrow \eta_t \\ M_{-t} & \xrightarrow{F} & N_{-t}. \end{array}$$

Proof. We know from the property of integral curves that for any $p \in M$, the curve $F \circ \theta^{(p)}$ is an integral curve of Y starting at $F \circ \theta^{(p)}(0) = F(p)$. Thus, by uniqueness of integral curves, the maximal integral curve $\eta^{(F(p))}$ must be defined at least on the interval $\mathcal{D}^{(p)}$, and $F \circ \theta^{(p)} = \eta^{(F(p))}$ on that interval. This means that

$$p \in M_t \Rightarrow t \in \mathcal{D}^{(p)} \Rightarrow t \in \mathcal{D}^{(F(p))} \Rightarrow F(p) \in N_t,$$

which is equivalent to $F(M_t) \subseteq N_t$, and

$$F(\theta^{(p)}(t)) = \eta^{(F(p))}(t) \quad \text{for all } t \in \mathcal{D}^{(p)},$$

which is equivalent to $\eta_t \circ F(p) = F \circ \theta_t(p)$ for all $p \in M_t$. □

As a corollary, we have that if $F : M \to N$ is a diffeomorphism and θ is the flow of a vector field $X \in \mathfrak{X}(M)$ then the flow of F_*X is $\eta_t = F \circ \theta_t \circ F^{-1}$, with domain $N_t = F(M_t)$ for each $t \in \mathbb{R}$.

9.3 Complete Vector Fields

Definition 9.18 (Complete vector fields)**.** We say that a smooth vector field is ***complete*** if it generates a global flow, or equivalently if each of its maximal integral curves is defined for all $t \in \mathbb{R}$.

It is not always easy to determine by looking at a vector field whether it is complete or not. If you can solve the ODE explicitly to find all of the integral curves, and they all exist for all time, then the vector field is complete. On the other hand, if you can find a single integral curve that cannot be extended to all of $\mathbb{R}$, as we did for the vector fields of examples above, then it is not complete. However, it is often impossible to solve the ODE explicitly, so it is useful to have some general criteria for determining when a vector field is complete.

We will show below that all compactly supported smooth vector fields, and therefore all smooth vector fields on a compact manifold, are complete. The proof will be based on the following lemma.

Lemma 9.19. *Let V be a smooth vector field on a smooth manifold M, and let θ be its flow. Suppose there is a positive number ε such that for every $p \in M$, the domain of $\theta^{(p)}$ contains $(-\varepsilon, \varepsilon)$. Then V is complete.*

Remark 9.20. What is this lemma actually saying? It is saying that if an integral curve of a vector field stops existing then the vector field must be "blowing up" in finite time. In other words, if an integral curve of a vector field can survive for even a short uniform time for every point in the manifold then it must exist for all time. Intuitively, the hypothesis that $\theta^{(p)}$ contains $(-\varepsilon, \varepsilon)$ for all p should explain that you can keep on applying the translation lemma and extending the existence time for the integral curve.

Proof. We prove it by contradiction. The basic idea is to use the translation property of integral curves. Suppose for the sake of contradiction that for some $p \in M$, the domain $\mathcal{D}^{(p)}$ of $\theta^{(p)}$ is bounded above. (A similar proof works if it is bounded below.) Let $b = \sup \mathcal{D}^{(p)}$, let t_0 be a positive number such that $b - \varepsilon < t_0 < b$, and let $q = \theta^{(p)}(t_0)$. The hypothesis implies that $\theta^{(q)}(t)$ is defined at least for $t \in (-\varepsilon, \varepsilon)$. Define a curve $\gamma\colon (-\varepsilon, t_0 + \varepsilon) \to M$ by

$$\gamma(t) = \begin{cases} \theta^{(p)}(t), & -\varepsilon < t < b, \\ \theta^{(q)}(t - t_0), & t_0 - \varepsilon < t < t_0 + \varepsilon. \end{cases}$$

These two definitions agree where they overlap, because $\theta^{(q)}(t-t_0) = \theta_{t-t_0}(q) = \theta_{t-t_0} \circ \theta_{t_0}(p) = \theta_t(p) = \theta^{(p)}(t)$ by the group law for θ. By the translation property, γ is an integral curve starting at p. Since $t_0 + \varepsilon > b$, this means that γ starting from p is existing beyond b, which is a contradiction to the definition of b. Thus, γ exists for all $t \in \mathbb{R}$ and hence V is a complete vector field. □

Theorem 9.21. *Every compactly supported smooth vector field on a smooth manifold is complete. Thus, on a compact smooth manifold, every smooth vector field is complete.*

Proof. Suppose V is a compactly supported vector field on a smooth manifold M, and let $K = \operatorname{supp} V$. For each $p \in K$, there is a neighbourhood U_p of p and a positive number ε_p such that the flow of V is defined at least on $(-\varepsilon_p, \varepsilon_p) \times U_p$. By compactness, finitely many such sets $U_{p_1}, \ldots, U_{p_k}$ cover K. With $\varepsilon = \min\{\varepsilon_{p_1}, \ldots, \varepsilon_{p_k}\}$, it follows that every maximal integral curve starting in K is defined at least on $(-\varepsilon, \varepsilon)$. Since $V \equiv 0$ outside of K, every integral curve starting in $M \setminus K$ is constant and thus can be defined on all of $\mathbb{R}$. Thus the hypotheses of Lemma 9.19 are satisfied, so V is complete. □

10. Lie Derivatives

We know how to differentiate real-valued functions in the direction of a vector field on a manifold. Indeed, a tangent vector $v \in T_pM$ is by definition an operator that acts on a smooth function f to give a number vf that we interpret as a directional derivative of f at p. How can we make sense of directional derivatives of a vector field in the direction of a given vector? Let us start from the familiar case of Euclidean spaces. We can define the directional derivative of a smooth vector field W in the direction of a vector $v \in T_p\mathbb{R}^n$. It is the vector

$$D_vW(p) = \left.\frac{d}{dt}\right|_{t=0} W_{p+tv} = \lim_{t\to 0} \frac{W_{p+tv} - W_p}{t}. \tag{10.1}$$

An easy calculation shows that $D_vW(p)$ can be evaluated by applying D_v to each component of W separately:

$$D_vW(p) = D_vW^i(p) \left.\frac{\partial}{\partial x^i}\right|_p .$$

Unfortunately, this definition is heavily dependent upon the fact that $\mathbb{R}^n$ is a vector space, so that the tangent vectors W_{p+tv} and W_p can both be viewed as elements of $\mathbb{R}^n$. If we want to mimic (10.1) on a manifold, we can try to replace $p + tv$ by a curve $\gamma(t)$ that starts at p and whose initial velocity is v. But even with this substitution, the difference quotient still makes no sense because $W_{\gamma(t)}$ and $W_{\gamma(0)}$ are elements of the two different vector spaces $T_{\gamma(t)}M$ and $T_{\gamma(0)}M$, respectively. It was not a problem in the Euclidean space because there is a canonical identification of each tangent space with $\mathbb{R}^n$ itself; but on a manifold there is no such identification. Thus there is no coordinate-independent way to make sense of the directional derivative of W in the direction of a vector v.

We fix this problem by replacing the vector $v \in T_pM$ with a *vector field* $V \in \mathfrak{X}(M)$, so we can use the flow of V to push values of W back to p and then differentiate.

Definition 10.1. Suppose M is a smooth manifold, V is a smooth vector field on M, and θ is the flow of V. For any smooth vector field W on M, define a vector field on M, denoted by $\mathcal{L}_VW$ and called the ***Lie derivative of W with respect to V***, by

$$\begin{aligned}(\mathcal{L}_VW)_p &= \left.\frac{d}{dt}\right|_{t=0} d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big) \\ &= \lim_{t\to 0} \frac{d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big) - W_p}{t},\end{aligned} \tag{10.2}$$

provided the derivative exists.

For small $t \neq 0$, at least the difference quotient makes sense: θ_t is defined in a neighbourhood of p, and θ_{-t} is the inverse of θ_t, so both $d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big)$ and W_p are elements of T_pM.

Lemma 10.2. *Suppose M is a smooth manifold and $V, W \in \mathfrak{X}(M)$. Then $(\mathcal{L}_VW)_p$ exists for every $p \in M$, and $\mathcal{L}_VW$ is a smooth vector field.*

Proof. We check that the component functions of the vector field $\mathcal{L}_VW$ are smooth in any coordinate chart. Let θ be the flow of V and for an arbitrary $p \in M$, let $(U, (x^i))$ be a smooth chart containing p. Choose an open interval J_0 containing 0 and an open subset $U_0 \subseteq U$ containing p such that θ maps $J_0 \times U_0$ into U. We can always do that using continuity of the flow θ. For $(t, x) \in J_0 \times U_0$, write the component functions of θ as $\big(\theta^1(t, x), \ldots, \theta^n(t, x)\big)$. Then for any $(t, x) \in J_0 \times U_0$, the matrix of $d(\theta_{-t})_{\theta_t(x)} : T_{\theta_t(x)}M \to T_xM$ is

$$\left[\frac{\partial\theta^i}{\partial x^j}(-t, \theta(t, x))\right]_{ij},$$

and hence,

$$d(\theta_{-t})_{\theta_t(x)}\big(W_{\theta_t(x)}\big) = \frac{\partial\theta^i}{\partial x^j}(-t, \theta(t, x))\, W^j(\theta(t, x)) \left.\frac{\partial}{\partial x^i}\right|_x .$$

Since θ^i and W^j are smooth functions, the coefficient of $\frac{\partial}{\partial x^i}|_x$ depends smoothly on (t, x). It follows that $(\mathcal{L}_VW)_x$, which is obtained by taking the derivative of this expression with respect to t and setting $t = 0$, exists for each $x \in U_0$ and depends smoothly on x. □

While equation 10.2 explains exactly how to make sense of the derivative of a vector field in the direction of another, it is not very useful for explicit computations, because typically the flow is difficult or impossible to write down explicitly. Fortunately, there is a simple formula for computing the Lie derivative without explicitly finding the flow.

Theorem 10.3. *If M is a smooth manifold and $X, Y \in \mathfrak{X}(M)$, then $\mathcal{L}_X Y = [X, Y]$.*

Proof. Let θ be the flow of X and let $p \in M$. We first note that (10.2) is equivalent to the expression

$$(\mathcal{L}_X Y)_p = \frac{d}{dt}(\theta_t^* Y)_p|_{t=0}, \quad \text{where } (\theta_t^* Y)_p = (\theta_t^{-1})_{*\theta_t(p)} Y_{\theta_t(p)}.$$

Recall also that for any $f \in C^\infty(M)$, $(Xf)_p = \frac{d}{dt}(\theta_t^* f)_p|_{t=0}$ where $\theta_t^* f = f \circ \theta_t$ and similarly we have $(\theta_t^* Y f)_p = (\theta_t^* Y)_p(\theta_t^* f)$. We now use all this and compute. We have

$$(X(Yf))_p = \lim_{t\to 0} \frac{(\theta_t^* Y f)_p - (Yf)_p}{t} = \lim_{t\to 0} \frac{(\theta_t^* Y)_p(\theta_t^* f) - (Yf)_p}{t}.$$

We add and subtract $(\theta_t^* Y)_p f$ to get

$$\begin{aligned}(X(Yf))_p &= \lim_{t\to 0} \frac{(\theta_t^* Y)_p(\theta_t^* f) - (\theta_t^* Y)_p f + (\theta_t^* Y)_p f - Y_p f}{t} \\ &= \lim_{t\to 0}(\theta_t^* Y)_p \frac{(\theta_t^* f) - f}{t} + \lim_{t\to 0} \frac{(\theta_t^* Y)_p - Y_p}{t} f = Y_p X f + (\mathcal{L}_X Y)_p.\end{aligned}$$

Thus, we get that $XY = YX + \mathcal{L}_X Y$.

□

We now have a geometric interpretation of the Lie bracket of two vector fields: it is the directional derivative of the second vector field along the flow of the first. We immediately get the following.

Proposition 10.4. Suppose M is a smooth manifold and $V, W, X \in \mathfrak{X}(M)$.

1. $\mathcal{L}_V W = -\mathcal{L}_W V$.
2. $\mathcal{L}_V [W, X] = [\mathcal{L}_V W, X] + [W, \mathcal{L}_V X]$.
3. $\mathcal{L}_{[V,W]} X = \mathcal{L}_V \mathcal{L}_W X - \mathcal{L}_W \mathcal{L}_V X$.
4. *If $g \in C^\infty(M)$, then $\mathcal{L}_V(gW) = (Vg)W + g\mathcal{L}_V W$.*
5. *If $F : M \to N$ is a diffeomorphism, then $F_*(\mathcal{L}_V X) = \mathcal{L}_{F_* V} F_* X$.*

Notice that Proposition 10.4 (d) is immediate from Proposition 8.19 (d) and it is of this form because the Lie bracket $[fV, gW]$ can be thought of as the Lie derivative $\mathcal{L}_{fV}(gW)$, it satisfies a product rule in g and W; and because it can also be thought of as $-\mathcal{L}_{gW}(fV)$, it satisfies a product rule in f and V as well.

If V and W are vector fields on M and θ is the flow of V, the Lie derivative $(\mathcal{L}_V W)_p$, by definition, expresses the t-derivative of the time-dependent vector $d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big) \in T_p M$ at $t = 0$. The next proposition shows how it can also be used to compute the derivative of this expression at other times.

Proposition 10.5. Suppose M is a smooth manifold and $V, W \in \mathfrak{X}(M)$. Let θ be the flow of V. For any (t_0, p) in the domain of θ,

$$\left.\frac{d}{dt}\right|_{t=t_0} d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big) = d(\theta_{-t_0})\big((\mathcal{L}_V W)_{\theta_{t_0}(p)}\big). \tag{10.3}$$

Proof. Let $p \in M$ be arbitrary, let $\mathcal{D}^{(p)} \subseteq \mathbb{R}$ denote the domain of the integral curve $\theta^{(p)}$, and consider the map $X : \mathcal{D}^{(p)} \to T_p M$ given by $X(t) = d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big)$. The same idea as in the proof of Lemma 10.2 shows that X is a smooth curve in the vector space T_pM. Making the change of variables $t = t_0 + s$, we obtain

$$\begin{aligned} X'(t_0) &= \frac{d}{ds}\bigg|_{s=0} X(t_0+s) = \frac{d}{ds}\bigg|_{s=0} d(\theta_{-t_0-s})\big(W_{\theta_{s+t_0}(p)}\big) \\ &= \frac{d}{ds}\bigg|_{s=0} d(\theta_{-t_0}) \circ d(\theta_{-s})\big(W_{\theta_s(\theta_{t_0}(p))}\big) \\ &= d(\theta_{-t_0}) \left(\frac{d}{ds}\bigg|_{s=0} d(\theta_{-s})\big(W_{\theta_s(\theta_{t_0}(p))}\big) \right). \end{aligned}$$

The last equality follows because $d(\theta_{-t_0}) : T_{\theta_{t_0}(p)}M \to T_pM$ is a linear map that is independent of s. By definition of the Lie derivative, this last expression is equal to the right-hand side of (10.3). □

10.1 Commuting Vector Fields

Definition 10.6. Let M be a smooth manifold and $V, W \in \mathfrak{X}(M)$. We say that V ***and*** W ***commute*** if $VWf = WVf$ for every smooth function f, or equivalently if $[V, W] \equiv 0$.

If θ is a smooth flow, a vector field W is said to be ***invariant under*** θ if W is θ_t-related to itself for each t or equivalently that $d(\theta_t)_p(W_p) = W_{\theta_t(p)}$ for all (t, p) in the domain of θ.

The next proposition shows that these two concepts are intimately related.

Proposition 10.7. *For smooth vector fields V and W on a smooth manifold M, the following are equivalent:*

(a) V and W commute.

(b) W is invariant under the flow of V.

(c) V is invariant under the flow of W.

As a result, every smooth vector field is invariant under its own flow.

Proof. Suppose $V, W \in \mathfrak{X}(M)$, and let θ denote the flow of V. We prove (b) $\implies$ (a). If (b) holds, then $W_{\theta_t(p)} = d(\theta_t)_p(W_p)$ whenever (t, p) is in the domain of θ. Applying $d(\theta_{-t})_{\theta_t(p)}$ to both sides, we conclude that $d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big) = W_p$, which by the definition of the Lie derivative or (10.2) implies $[V, W] = \mathcal{L}_V W = 0$. The same argument shows that (c) implies (a).

To prove that (a) implies (b), assume that $[V, W] = \mathcal{L}_V W = 0$. Let $p \in M$ be arbitrary, and let $X(t) = d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big)$ for $t \in \mathcal{D}^{(p)}$. Proposition 10.5 shows that $X'(t) \equiv 0$. Since $X(0) = W_p$, this implies that $X(t) = W_p$ for all $t \in \mathcal{D}^{(p)}$, and applying $d(\theta_t)_p$ to both sides yields the identity that says W is invariant under θ. Similarly (a) implies (c).

□

We now relate commuting vector fields in terms of the relationship between their respective flows. More precisely, the next theorem will prove that two vector fields commute if and only if their flows commute. But what does two flows commuting with each other means?

Suppose V and W are smooth vector fields on M, and let θ and ψ be their respective flows. If V and W are complete vector fields and hence θ and ψ are global flows, then their flows commute means

$$\theta_t \circ \psi_s = \psi_s \circ \theta_t \qquad \text{for all } s, t \in \mathbb{R}.$$

However, if either V or W is not complete, the most we can hope for is that this equation holds for all s and t such that both sides are defined.

Unfortunately, even when the vector fields commute, their flows might not commute in this naive sense, because there are examples of commuting vector fields V and W and particular choices of t, s, and p for which both $\theta_t \circ \psi_s(p)$ and $\psi_s \circ \theta_t(p)$ are defined, but they are not equal.

Exercise 10.8. Consider $\mathbb{R}^3$ and consider the vector fields

$$V = \frac{\partial}{\partial x} - \frac{y}{x^2+y^2}\frac{\partial}{\partial z}, \quad W = \frac{\partial}{\partial y} + \frac{x}{x^2+y^2}\frac{\partial}{\partial z}.$$

Let θ and ψ be the flow of V and W respectively. Show that V and W commute but there exist $p \in \mathbb{R}^3$ and $s, t \in \mathbb{R}$ such that $\theta_t \circ \psi_s(p)$ and $\psi_s \circ \theta_t(p)$ both exist but are not equal. (**Hint:** for instance, $p = (1, -1, 0)$ and $s = 2, t = -2$ does the job. Maybe write the equation for the integral curves.)

Here is the problem: if $\theta_t \circ \psi_s(p)$ is defined for $t = t_0$ and $s = s_0$, then by the properties of flow domains, it must be defined for all t in some open interval containing 0 and t_0, but the analogous statement need not be true of s - there might be values of s between 0 and s_0 for which the integral curve of V starting at $\psi_s(p)$ does not extend all the way to $t = t_0$.

Thus we make the following definition.

Definition 10.9. If θ and ψ are flows on M, we say that θ ***and*** ψ ***commute*** if the following condition holds for every $p \in M$: whenever J and K are open intervals containing 0 such that one of the expressions $\theta_t \circ \psi_s(p)$ or $\psi_s \circ \theta_t(p)$ is defined for all $(s, t) \in J \times K$, both are defined and they are equal. For global flows, this is the same as saying that $\theta_t \circ \psi_s = \psi_s \circ \theta_t$ for all s and t.

Here is the main result for commuting flows.

Theorem 10.10. *Smooth vector fields commute if and only if their flows commute.*

Proof. Let V and W be smooth vector fields on a smooth manifold M, and let θ and ψ denote their respective flows. Assume first that V and W commute. Suppose that $p \in M$, and J and K are open intervals containing 0 such that $\psi_s \circ \theta_t(p)$ is defined for all $(s, t) \in J \times K$. By Proposition 10.7, the hypothesis implies that V is invariant under ψ. Fix any $s \in J$, and consider the curve $\gamma : K \to M$ defined by $\gamma(t) = \psi_s \circ \theta_t(p) = \psi_s\big(\theta^{(p)}(t)\big)$. This curve satisfies $\gamma(0) = \psi_s(p)$, and its velocity at $t \in K$ is

$$\gamma'(t) = \frac{d}{dt}\big(\psi_s(\theta^{(p)}(t))\big) = d(\psi_s)\big(\theta^{(p)\prime}(t)\big) = d(\psi_s)(V_{\theta^{(p)}(t)}) = V_{\gamma(t)}.$$

Thus, γ is an integral curve of V starting at $\psi_s(p)$. By uniqueness, therefore,

$$\gamma(t) = \theta^{\psi_s(p)}(t) = \theta_t\big(\psi_s(p)\big).$$

This proves that θ and ψ commute.

Conversely, assume that the flows commute, and let $p \in M$. If $\varepsilon > 0$ is chosen small enough that $\psi_s \circ \theta_t(p)$ is defined whenever $|s| < \varepsilon$ and $|t| < \varepsilon$, then the hypothesis guarantees that $\psi_s \circ \theta_t(p) = \theta_t \circ \psi_s(p)$ for all such s and t. This can be rewritten in the form

$$\psi^{\theta_t(p)}(s) = \theta_t\big(\psi^{(p)}(s)\big).$$

Differentiating this relation with respect to s, we get

$$W_{\theta_t(p)} = \frac{d}{ds}\bigg|_{s=0} \psi^{\theta_t(p)}(s) = \frac{d}{ds}\bigg|_{s=0} \theta_t\big(\psi^{(p)}(s)\big) = d(\theta_t)_p(W_p).$$

Applying $d(\theta_{-t})_{\theta_t(p)}$ to both sides, we conclude

$$d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big) = W_p.$$

Differentiating with respect to t and applying the definition of the Lie derivative shows that $(\mathcal{L}_V W)_p = 0$. □