9.2 Flows of vector fields

Let M be a smooth manifold and $V \in \mathfrak{X}(M)$, and suppose that for each point $p \in M$, V has a unique integral curve starting at p and defined for all $t \in \mathbb{R}$, which we denote by $\theta^{(p)} \colon \mathbb{R} \to M$. (We don't know whether this is always possible or not.) For each $t \in \mathbb{R}$, we can define a map $\theta_t \colon M \to M$ by sending each $p \in M$ to the point obtained by following for time t the integral curve starting at p:

$$\theta_t(p) = \theta^{(p)}(t).$$

Each map θ_t "slides" the manifold along the integral curves for time t. Then $t \mapsto \theta^{(p)}(t+s)$ is an integral curve of V starting at $q = \theta^{(p)}(s)$ and since we are assuming uniqueness of integral curves, $\theta^{(q)}(t) = \theta^{(p)}(t+s)$. When we translate this into a statement about the maps θ_t, it becomes

$$\theta_t \circ \theta_s(p) = \theta_{t+s}(p).$$

Together with the equation $\theta_0(p) = \theta^{(p)}(0) = p$, which holds by definition, this implies that the map

$$\theta \colon \mathbb{R} \times M \to M \text{ is an action of the additive group } \mathbb{R} \text{ on } M.$$

Definition 9.6. A **global flow** on M which is also sometimes called a **one-parameter group action** is a continuous left $\mathbb{R}$-action on M; that is, a continuous map $\theta \colon \mathbb{R} \times M \to M$ satisfying the following properties for all $s, t \in \mathbb{R}$ and $p \in M$:

$$\theta(t, \theta(s,p)) = \theta(t+s, p), \qquad \theta(0,p) = p. \tag{9.3}$$

Given a global flow θ on M, we have the following two collections of maps:

- For each $t \in \mathbb{R}$, define a continuous map $\theta_t \colon M \to M$ by

  $$\theta_t(p) = \theta(t,p).$$

  which satisfy

  $$\theta_t \circ \theta_s = \theta_{t+s}, \qquad \theta_0 = \mathrm{Id}_M. \tag{9.4}$$

  As is the case for any continuous group action, each map $\theta_t \colon M \to M$ is a homeomorphism, and if the flow is smooth, θ_t is a diffeomorphism.

- For each $p \in M$, define a curve $\theta^{(p)} \colon \mathbb{R} \to M$ by

  $$\theta^{(p)}(t) = \theta(t,p).$$

  The image of this curve is **the orbit of p under the group action.**

The next proposition shows that every smooth global flow is derived from the integral curves of some smooth vector field in precisely the way we described above. Before stating the result let us make the following

Definition 9.7. If $\theta \colon \mathbb{R} \times M \to M$ is a smooth global flow, for each $p \in M$ we define a tangent vector $V_p \in T_pM$ by

$$V_p = \theta^{(p)\prime}(0). \tag{9.5}$$

The assignment $p \mapsto V_p$ is a vector field on M, which is called the **infinitesimal generator of θ**.

Lemma 9.8. *Let $\theta \colon \mathbb{R} \times M \to M$ be a smooth global flow on a smooth manifold M. The infinitesimal generator V of θ is a smooth vector field on M, and each curve $\theta^{(p)}$ is an integral curve of V.*

Proof. To show that V is smooth we show that Vf is smooth for every $f \in C^\infty(U)$, $U \subseteq M$. For any such f and any $p \in U$, just note that

$$Vf(p) = V_p f = \theta^{(p)\prime}(0) f = \left.\frac{d}{dt}\right|_{t=0} f(\theta^{(p)}(t)) = \left.\frac{\partial}{\partial t}\right|_{(0,p)} f(\theta(t,p)).$$

Because $f(\theta(t,p))$ is a smooth function of (t,p) by composition, so is its partial derivative with respect to t. Thus, $Vf(p)$ depends smoothly on p, so V is smooth.

Next we need to show that $\theta^{(p)}$ is an integral curve of V, which means that $\theta^{(p)\prime}(t) = V_{\theta^{(p)}(t)}$ for all $p \in M$ and all $t \in \mathbb{R}$. Let $t_0 \in \mathbb{R}$ be arbitrary, and set $q = \theta^{(p)}(t_0) = \theta_{t_0}(p)$, so what we have to show is $\theta^{(p)\prime}(t_0) = V_q$. By the group law, for all t,

$$\theta^{(q)}(t) = \theta_t(q) = \theta_t(\theta_{t_0}(p)) = \theta_{t+t_0}(p) = \theta^{(p)}(t+t_0).$$

Therefore, for any smooth real-valued function f defined in a neighbourhood of q,

$$\begin{aligned} V_q f = \theta^{(q)\prime}(0) f = \frac{d}{dt}\bigg|_{t=0} f(\theta^{(q)}(t)) = \frac{d}{dt}\bigg|_{t=0} f(\theta^{(p)}(t+t_0)) \\ = \theta^{(p)\prime}(t_0) f, \end{aligned}$$

which was to be shown. □

Example 9.9 ((Global Flows).). The two vector fields on $\mathbb{R}^2$ described in Examples 9.2 and 9.3 both had integral curves defined for all $t \in \mathbb{R}$, so they generate global flows. Using the results of that example, we can write down the flows explicitly.

(a) The flow of $V = \partial/\partial x$ in $\mathbb{R}^2$ is the map $\tau \colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\tau_t(x,y) = (x+t, y).$$

For each nonzero $t \in \mathbb{R}$, τ_t translates the plane to the right ($t > 0$) or left ($t < 0$) by a distance $|t|$.

(b) The flow of $W = x\,\partial/\partial y - y\,\partial/\partial x$ is the map $\theta \colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\theta_t(x,y) = (x\cos t - y\sin t, x\sin t + y\cos t).$$

For each $t \in \mathbb{R}$, θ_t rotates the plane through an angle t about the origin.

Thus,

infinitesimal generators of global flows are smooth vector fields

A natural question arises: Is the converse of the above statement true, i.e., given a smooth $V \in \mathfrak{X}(M)$, must it be an infinitesimal generator of some global flow? Let's see some examples with smooth vector fields whose integral curves are not defined for all $t \in \mathbb{R}$.

Example 9.10. Let $M = \mathbb{R}^2 \setminus \{0\}$ with standard coordinates (x,y), and let V be the vector field $\frac{\partial}{\partial x}$ on M. The unique integral curve of V starting at $(-1,0) \in M$ is $\gamma(t) = (t-1, 0)$. However, in this case, γ cannot be extended continuously past $t = 1$.

Example 9.11. Let M be all of $\mathbb{R}^2$ and let $W = x^2 \frac{\partial}{\partial x}$. The unique integral curve of W starting at $(1,0)$ is

$$\gamma(t) = \left(\frac{1}{1-t}, 0\right).$$

This curve also cannot be extended past $t = 1$, because its x-coordinate is unbounded as $t \nearrow 1$.

Thus, not every smooth vector field V gives rise to global flows by being their infinitesimal generator.

Definition 9.12. If M is a manifold, a **flow domain** for M is an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ with the property that for each $p \in M$, the set

$$\mathcal{D}^{(p)} = \{t \in \mathbb{R} : (t,p) \in \mathcal{D}\}$$

is an open interval containing 0.

A **flow** on M is a continuous map $\theta \colon \mathcal{D} \to M$, where $\mathcal{D} \subseteq \mathbb{R} \times M$ is a flow domain, that satisfies the following group laws: for all $p \in M$,

$$\theta(0,p) = p, \tag{9.6}$$

and for all $s \in \mathcal{D}^{(p)}$ and $t \in \mathcal{D}^{(\theta(s,p))}$ such that $s + t \in \mathcal{D}^{(p)}$,

$$\theta(t, \theta(s,p)) = \theta(t+s, p). \tag{9.7}$$

The flow θ is also sometimes called a **local flow** or a **local one-parameter group action**.

If θ is smooth, the ***infinitesimal generator of θ*** is defined by $V_p = \theta^{(p)\prime}(0)$.