9. Integral curves and flows of vector fields

9.1 Integral Curves

Suppose M is a smooth manifold. If $\gamma\colon J \to M$ is a smooth curve, then for each $t \in J$, the velocity vector $\gamma'(t)$ is a vector in $T_{\gamma(t)}M$. What about the reverse process? That is, given $V \in \mathfrak{X}(M)$, can we come up with a curve γ such that the tangent vector is the corresponding image of the vector field?

Definition 9.1. Let V is a vector field on M. An **integral curve** of V is a differentiable curve $\gamma\colon J \to M$ whose velocity at each point is equal to the value of V at that point:

$$\gamma'(t) = V_{\gamma(t)} \quad \text{for all } t \in J. \tag{9.1}$$

If $0 \in J$, the point $\gamma(0)$ is called the **starting point of γ**.

Before dwelling further in the theory, let us see some simple examples.

Example 9.2. Let (x, y) be standard coordinates on $\mathbb{R}^2$, and let $V = \frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2)$. Then

$$\gamma(t) = (a + t, b), \quad a, b \text{ constants}$$

is an integral curve for V. Things to notice is that given any $(p, q) \in \mathbb{R}^2$ $\exists!$ integral curve of V starting from (p, q) and the images of two integral curves are either identical or disjoint. It is easy to check that the integral curves of V are precisely the straight lines parallel to the x-axis.

Example 9.3. Let $W = x\,\frac{\partial}{\partial y} - y\,\partial\partial x \in \mathfrak{X}(\mathbb{R}^2)$. If $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ is a smooth curve, written in standard coordinates as $\gamma(t) = (x(t), y(t))$, then the condition $\gamma'(t) = W_{\gamma(t)}$ for γ to be an integral curve translates to

$$x'(t)\,\frac{\partial}{\partial x}\bigg|_{\gamma(t)} + y'(t)\,\frac{\partial}{\partial y}\bigg|_{\gamma(t)} = x(t)\,\frac{\partial}{\partial y}\bigg|_{\gamma(t)} - y(t)\,\frac{\partial}{\partial x}\bigg|_{\gamma(t)}.$$

Comparing the components of these vectors, we see that this is equivalent to the system of ordinary differential equations

$$\begin{aligned} x'(t) &= -y(t), \\ y'(t) &= x(t). \end{aligned}$$

These equations have the solutions

$$x(t) = a\cos t - b\sin t, \qquad y(t) = a\sin t + b\cos t,$$

for arbitrary constants a and b, and thus each curve of the form $\gamma(t) = (a\cos t - b\sin t, a\sin t + b\cos t)$ is an integral curve of W starting at (a, b). When $(a, b) = (0, 0)$, this is the constant curve $\gamma(t) \equiv (0, 0)$; otherwise, it is a circle traversed counterclockwise. Thus, given any $(a, b) \in \mathbb{R}^2$, we see once again that there is a unique integral curve starting at each point $(a, b) \in \mathbb{R}^2$, and the images of the various integral curves are either identical or disjoint.

From this example we see that given $V \in \mathfrak{X}(M)$, finding integral curve of V boils down to solving a system of ordinary differential equations in a smooth chart. More precisely, suppose V is a smooth vector field on M and $\gamma\colon J \to M$ is a smooth curve. On a smooth coordinate domain $U \subseteq M$, we can write γ in local coordinates as $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$. Then (9.1) for γ to be an integral curve of V can be written

$$\dot\gamma^i(t)\,\frac{\partial}{\partial x^i}\bigg|_{\gamma(t)} = V^i(\gamma(t))\,\frac{\partial}{\partial x^i}\bigg|_{\gamma(t)},$$

which reduces to the following autonomous system of ordinary differential equations (ODEs):

$$\begin{aligned} \dot\gamma^1(t) &= V^1(\gamma^1(t), \dots, \gamma^n(t)), \\ &\vdots \\ \dot\gamma^n(t) &= V^n(\gamma^1(t), \dots, \gamma^n(t)). \end{aligned} \tag{9.2}$$

This means that to guarantee the existence of integral curves, we would need to solve an ordinary differential equation (ODE). Thus, we first recall the following fundamental result on ODEs.

Theorem 9.4 (Fundamental theorem for ODEs). *Suppose $U \subset \mathbb{R}^n$ is open and $V : U \to \mathbb{R}^n$ is a smooth vector-valued function. Consider the system of ODE*

$$\dot{\gamma}^i(t) = V^i(\gamma^1(t), \ldots, \gamma^n(t)),\ i = 1, \ldots, n,$$
$$\gamma^i(t_0) = c^i,\ i = 1, \ldots, n.$$

with $c = (c^1, \ldots, c^n) \in U$. Then

1. *(existence) $\forall t_0 \in \mathbb{R}$ and $x_0 \in U$ there exists an open interval $J_0 \ni t_0$ and an open subset $U_0 \subset U$ with $x_0 \in U_0$ such that for all $c \in U_0$ there exists a C^1-curve $\gamma : J_0 \to U$ that solves the ODE.*
2. *(Uniqueness) Any two solutions γ_1 and γ_2 of the above ODE are same on their common domain.*
3. *The map $\theta : J_0 \times U_0 \to U$ given by $\theta(t, x) = \gamma(t)$ where the γ is the unique solution above and $\gamma(t_0) = x$, is a smooth map.*

Using this theorem we have the following simple result.

Proposition 9.5. *Let V be a smooth vector field on a smooth manifold M. For each point $p \in M$, there exist $\varepsilon > 0$ and a smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ that is an integral curve of V starting at p.* □

We notice some elementary properties of integral curves.

- If $a \in \mathbb{R}$ and $\gamma\colon J \to M$ is an integral curve of $V \in \mathfrak{X}(M)$, then the curve $\widetilde{\gamma}\colon \widetilde{J} \to M$ defined by $\widetilde{\gamma}(t) = \gamma(at)$ is an integral curve of the vector field aV, where $\widetilde{J} = \{t : at \in J\}$.

  Proof. We can examine the action of $\widetilde{\gamma}'(t)$ on a smooth real-valued function f defined in a neighbourhood of a point $\widetilde{\gamma}(t_0)$. By the chain rule and the fact that γ is an integral curve of V,

  $$\widetilde{\gamma}'(t_0)f = \frac{d}{dt}\bigg|_{t=t_0} (f \circ \widetilde{\gamma})(t) = \frac{d}{dt}\bigg|_{t=t_0} (f \circ \gamma)(at)$$
  $$= a(f \circ \gamma)'(at_0) = a\gamma'(at_0)f = aV_{\widetilde{\gamma}(t_0)}f. \qquad \square$$

- Let V, M, J, and γ be as before. For any $b \in \mathbb{R}$, the curve $\widehat{\gamma}\colon \widehat{J} \to M$ defined by $\widehat{\gamma}(t) = \gamma(t + b)$ is also an integral curve of V, where $\widehat{J} = \{t : t + b \in J\}$.

- Suppose M and N are smooth manifolds and $F\colon M \to N$ is a smooth map. Then $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related if and only if F takes integral curves of X to integral curves of Y, meaning that for each integral curve γ of X, $F \circ \gamma$ is an integral curve of Y.

  Proof. Suppose first that X and Y are F-related, and $\gamma\colon J \to M$ is an integral curve of X. If we define $\sigma\colon J \to N$ by $\sigma = F \circ \gamma$, then

  $$\sigma'(t) = (F \circ \gamma)'(t) = dF_{\gamma(t)}(\gamma'(t)) = dF_{\gamma(t)}(X_{\gamma(t)}) = Y_{F(\gamma(t))} = Y_{\sigma(t)},$$

  so σ is an integral curve of Y.

  Conversely, suppose F takes integral curves of X to integral curves of Y. Given $p \in M$, let $\gamma\colon (-\varepsilon, \varepsilon) \to M$ be an integral curve of X starting at p. Since $F \circ \gamma$ is an integral curve of Y starting at $F(p)$, we have

  $$Y_{F(p)} = (F \circ \gamma)'(0) = dF_p(\gamma'(0)) = dF_p(X_p),$$

  which shows that X and Y are F-related. □