It is important to remember that for a given smooth map $F : M \to N$ and vector field $X \in \mathfrak{X}(M)$, there may not be *any* vector field on N that is F-related to X. There is one special case, however, in which there is always such a vector field, as the next lemma shows.

Lemma 8.12. *Suppose M and N are smooth manifolds and $F : M \to N$ is a diffeomorphism. For every $X \in \mathfrak{X}(M)$, there is a unique smooth vector field on N that is F-related to X.*

Proof. For $Y \in \mathfrak{X}(N)$ to be F-related to X means that $dF_p(X_p) = Y_{F(p)}$ for every $p \in M$. If F is a diffeomorphism, we *define* Y by

$$Y_q = dF_{F^{-1}(q)}\left(X_{F^{-1}(q)}\right).$$

It is clear that Y, so defined, is the unique vector field that is F-related to X. Note that $Y : N \to TN$ is the composition of the following smooth maps:

$$N \xrightarrow{F^{-1}} M \xrightarrow{X} TM \xrightarrow{dF} TN.$$

It follows that Y is smooth. □

We can now make the following definition.

Definition 8.13 (Pushforwards and Pullbacks)**.** Let $F : M \to N$ be a diffeomorphism between smooth manifolds M and N. We denote the unique vector field that is F-related to X by F_*X, and call it the **pushforward of X by F**. It is explicitly described by

$$(F_*X)_q = dF_{F^{-1}(q)}\left(X_{F^{-1}(q)}\right). \tag{8.6}$$

The **pullback of a vector field** $Y \in \mathfrak{X}(N)$ is denoted by F^*Y and is defined as

$$F^*Y = dF^{-1} \circ Y \circ F \in \mathfrak{X}(M). \tag{8.7}$$

As long as the inverse map F^{-1} can be computed explicitly, the pushforward of a vector field can be computed directly from the formula in (8.6).

Let us compute the pushforward of a vector field explicitly.

Let M and N be the following open submanifolds of $\mathbb{R}^2$:

$$M = \{(x, y) : y > 0 \text{ and } x + y > 0\},$$

$$N = \{(u, v) : u > 0 \text{ and } v > 0\},$$

and define $F : M \to N$ by $F(x, y) = (x + y, x/y + 1)$. Then F is a diffeomorphism because its inverse is easily computed: just solve $(u, v) = (x + y, x/y + 1)$ for x and y to obtain the formula $(x, y) = F^{-1}(u, v) = (u - u/v, u/v)$. Let us compute the pushforward F_*X, where X is the following smooth vector field on M:

$$X_{(x,y)} = y^2 \left.\frac{\partial}{\partial x}\right|_{(x,y)}.$$

The differential of F at a point $(x, y) \in M$ is represented by its Jacobian matrix,

$$DF(x, y) = \begin{pmatrix} 1 & 1 \\ \frac{1}{y} & -\frac{x}{y^2} \end{pmatrix},$$

and thus $dF_{F^{-1}(u,v)}$ is represented by the matrix

$$DF\left(u - \frac{u}{v}, \frac{u}{v}\right) = \begin{pmatrix} 1 & 1 \\ \frac{v}{u} & \frac{v-v^2}{u} \end{pmatrix}.$$

For any $(u, v) \in N$,

$$X_{F^{-1}(u,v)} = \frac{u^2}{v^2} \left.\frac{\partial}{\partial x}\right|_{F^{-1}(u,v)}.$$

Therefore, applying (8.6) with $p = (u, v)$ yields the formula for F_*X:

$$(F_*X)_{(u,v)} = \frac{u^2}{v^2} \left.\frac{\partial}{\partial u}\right|_{(u,v)} + \frac{u}{v} \left.\frac{\partial}{\partial v}\right|_{(u,v)}.$$

8.5 Vector Fields and Submanifolds

If $S \subseteq M$ is an embedded submanifold, a vector field X on M does not necessarily restrict to a vector field on S, because X_p may not lie in the subspace $T_pS \subseteq T_pM$ at a point $p \in S$. Given a point $p \in S$, a vector field X on M is said to be **tangent to** S **at** p if $X_p \in T_pS \subseteq T_pM$. It is **tangent to** S if it is tangent to S at every point of S.

Suppose $S \subseteq M$ is an embedded submanifold and Y is a smooth vector field on M. If there is a vector field $X \in \mathfrak{X}(S)$ that is ι-related to Y, where $\iota : S \hookrightarrow M$ is the inclusion map, then clearly Y is tangent to S, because $Y_p = d\iota_p(X_p)$ is in the image of $d\iota_p$ for each $p \in S$. The next proposition shows that the converse is true.

Proposition 8.14 (Restricting Vector Fields to Submanifolds)**.** *Let M be a smooth manifold, let $S \subseteq M$ be an embedded submanifold and let $\iota : S \hookrightarrow M$ denote the inclusion map. If $Y \in \mathfrak{X}(M)$ is tangent to S, then there is a unique smooth vector field on S, denoted by $Y|_S$, that is ι-related to Y.*

Proof. The fact that Y is tangent to S means by definition that Y_p is in the image of $d\iota_p$ for each p. Thus, for each p there is a vector $X_p \in T_pS$ such that $Y_p = d\iota_p(X_p)$. Since $d\iota_p$ is injective, X_p is unique, so this defines X as a vector field on S. If we can show that X is smooth, it is the unique vector field that is ι-related to Y. It suffices to show that it is smooth in a neighbourhood of each point.

Let p be any point in S. Since S is an embedded submanifold, let $(U, (x^i))$ be a slice chart in M centred at p, so that $S \cap U$ is the subset where $x^{k+1} = \cdots = x^n = 0$, and $(x^1, \ldots, x^k)$ form local coordinates for S in U. If $Y = Y^1 \frac{\partial}{\partial x^1} + \cdots + Y^n \frac{\partial}{\partial x^n}$ in these coordinates, it follows from our construction that X has the coordinate representation $Y^1 \frac{\partial}{\partial x^1} + \cdots + Y^k \frac{\partial}{\partial x^k}$, which is clearly smooth on U. □

8.6 Lie Brackets

In this section we introduce an important way of combining two smooth vector fields to obtain another vector field.

Let X and Y be smooth vector fields on a smooth manifold M. Given a smooth function $f : M \to \mathbb{R}$, we can apply X to f and obtain another smooth function Xf. In turn, we can apply Y to this function, and obtain yet another smooth function $YXf = Y(Xf)$. The operation $f \mapsto YXf$, however, does not in general satisfy the product rule and thus cannot be a vector field, as the following example shows.

Example 8.15. Define vector fields $X = \frac{\partial}{\partial x}$ and $Y = x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$, and let $f(x, y) = x$, $g(x, y) = y$. Then direct computation shows that $XY(fg) = 2x$, while $fXYg + gXYf = x$, so XY is not a derivation of $C^\infty(\mathbb{R}^2)$.

We can also apply the same two vector fields in the opposite order, obtaining a (usually different) function XYf. Applying both of these operators to f and subtracting, we obtain an operator

$$[X, Y] : C^\infty(M) \to C^\infty(M),$$

called the **Lie bracket of** X **and** Y, defined by

$$[X, Y]f = XYf - YXf.$$

The key fact is that this operator *is* a vector field.

Lemma 8.16. *The Lie bracket of any pair of smooth vector fields is a smooth vector field.*

Proof. It suffices to show that $[X, Y]$ is a derivation of $C^\infty(M)$. For arbitrary $f, g \in C^\infty(M)$, we compute

$$\begin{aligned}
[X, Y](fg) &= X(Y(fg)) - Y(X(fg)) \\
&= X(fYg + gYf) - Y(fXg + gXf) \\
&= Xf\,Yg + f\,XYg + Xg\,Yf + g\,XYf \\
&\quad - Yf\,Xg - f\,YXg - Yg\,Xf - g\,YXf \\
&= f\,XYg + g\,XYf - f\,YXg - g\,YXf \\
&= f[X, Y]g + g[X, Y]f.
\end{aligned}$$

□

The value of the vector field $[X, Y]$ at a point $p \in M$ is the derivation at p given by the formula

$$[X, Y]_p f = X_p(Yf) - Y_p(Xf). \tag{8.8}$$

However, this formula is of limited usefulness for computations, because it requires one to compute terms involving second derivatives of f that will always cancel each other out. The next proposition gives an extremely useful coordinate formula for the Lie bracket, in which the cancellations have already been accounted for.

Proposition 8.17 (Coordinate Formula for the Lie Bracket). *Let X, Y be smooth vector fields on a smooth manifold M and let $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$ be the coordinate expressions for X and Y in terms of some smooth local coordinates (x^i) for M. Then $[X, Y]$ has the following coordinate expression:*

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}, \tag{8.9}$$

or more concisely,

$$[X, Y] = (XY^j - YX^j) \frac{\partial}{\partial x^j}. \tag{8.10}$$

Proof. Because we know already that $[X, Y]$ is a smooth vector field, its action on a function is determined locally: $([X, Y]f)|_U = [X, Y](f|_U)$. Thus it suffices to compute in a single smooth chart, where we have

$$\begin{aligned}
[X, Y]f &= X^i \frac{\partial}{\partial x^i} \left(Y^j \frac{\partial f}{\partial x^j} \right) - Y^j \frac{\partial}{\partial x^j} \left(X^i \frac{\partial f}{\partial x^i} \right) \\
&= X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} + X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} - Y^j X^i \frac{\partial^2 f}{\partial x^j \partial x^i} \\
&= X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i},
\end{aligned}$$

where in the last step we have used the fact that mixed partial derivatives of a smooth function can be taken in any order. Interchanging the roles of the dummy indices i and j in the second term, we obtain (8.9). □

One trivial application of (8.9) is to compute the Lie brackets of the coordinate vector fields $(\partial/\partial x^i)$ in any smooth chart: because the component functions of the coordinate vector fields are all constants, it follows that

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] \equiv 0 \quad \text{for all } i \text{ and } j. \tag{8.11}$$

Notice that this also follows from the definition of the Lie bracket in (8.8), and is essentially a restatement of the fact that mixed partial derivatives of smooth functions commute. Here is a slightly less trivial computation.

Example 8.18. Define smooth vector fields $X, Y \in \mathfrak{X}(\mathbb{R}^3)$ by

$$X = x \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1) \frac{\partial}{\partial z},$$

$$Y = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}.$$

Then (8.10) gives us

$$\begin{aligned}
[X, Y] &= X(1) \frac{\partial}{\partial x} + X(y) \frac{\partial}{\partial z} - Y(x) \frac{\partial}{\partial x} - Y(1) \frac{\partial}{\partial y} - Y(x(y+1)) \frac{\partial}{\partial z} \\
&= 0 \frac{\partial}{\partial x} + 1 \frac{\partial}{\partial z} - 1 \frac{\partial}{\partial x} - 0 \frac{\partial}{\partial y} - (y+1) \frac{\partial}{\partial z} \\
&= -\frac{\partial}{\partial x} - y \frac{\partial}{\partial z},
\end{aligned}$$

which is indeed a vector field.

Let us now see some basic properties of the Lie bracket.

Proposition 8.19 (Properties of the Lie Bracket). *The Lie bracket satisfies the following identities for all* $X, Y, Z \in \mathfrak{X}(M)$*:*

(a) BILINEARITY: *For* $a, b \in \mathbb{R}$*,*

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$

(b) ANTISYMMETRY:

$$[X, Y] = -[Y, X].$$

(c) JACOBI IDENTITY:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

(d) For $f, g \in C^\infty(M)$*,*

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X. \tag{8.12}$$

Proof. You can check bilinearity and antisymmetry yourself. The proof of the Jacobi identity is just a computation:

$$\begin{aligned}
&[X, [Y, Z]]f + [Y, [Z, X]]f + [Z, [X, Y]]f \\
&\quad = X[Y, Z]f - [Y, Z]Xf + Y[Z, X]f \\
&\qquad - [Z, X]Yf + Z[X, Y]f - [X, Y]Zf \\
&\quad = XYZf - XZYf - YZXf + ZYXf + YZXf - YXZf \\
&\qquad - ZXYf + XZYf + ZXYf - ZYXf - XYZf + YXZf.
\end{aligned}$$

In this last expression all the terms cancel in pairs. Part (d) is again a direct computation from the definition of the Lie bracket. Let $h \in C^\infty(M)$. We have

$$\begin{aligned}
[fX, gY]h &= (fX)(gY)h - (gY)(fX)h \\
&= (fX)(gYh) - (gY)(fXh) \\
&= fg(XYh) + f(Xg)(Yh) - fg(YXh) - g(Yf)(Xh) \\
&= fg[X, Y]h + (fXg)Yh - (gYf)Xh.
\end{aligned}$$

□

The significance of part (d) of this proposition might not be evident at this point, but it will become clearer in the future lectures when we will talk about the *Lie derivative of a vector field in the direction of another vector field* where it will be shown to be equal to the Lie bracket and hence the Lie derivative will indeed be a derivation using part (d).

The next result relates the Lie bracket of F-related vector fields.

Proposition 8.20 (Naturality of the Lie Bracket). *Let* $F : M \to N$ *be a smooth map between manifolds and let* $X_1, X_2 \in \mathfrak{X}(M)$ *and* $Y_1, Y_2 \in \mathfrak{X}(N)$ *be vector fields such that* X_i *is* F*-related to* Y_i *for* $i = 1, 2$*. Then* $[X_1, X_2]$ *is* F*-related to* $[Y_1, Y_2]$*.*

Proof. Using (8.5) and the fact that X_i and Y_i are F-related,

$$X_1X_2(f \circ F) = X_1\big((Y_2f) \circ F\big) = (Y_1Y_2f) \circ F.$$

Similarly,

$$X_2X_1(f \circ F) = (Y_2Y_1f) \circ F.$$

Therefore,

$$\begin{aligned}
[X_1, X_2](f \circ F) &= X_1X_2(f \circ F) - X_2X_1(f \circ F) \\
&= (Y_1Y_2f) \circ F - (Y_2Y_1f) \circ F \\
&= \big([Y_1, Y_2]f\big) \circ F.
\end{aligned}$$

□

When applied in special cases, this result has the following important corollaries. First we consider the case in which the map is a diffeomorphism.

Corollary 8.21. *We have the following.*

1. *(Pushforwards of Lie Brackets) Suppose $F : M \to N$ is a diffeomorphism and $X_1, X_2 \in \mathfrak{X}(M)$. Then $F_*[X_1, X_2] = [F_*X_1, F_*X_2]$.*
2. *(Brackets of Vector Fields Tangent to Submanifolds) Let M be a smooth manifold and let S be an embedded submanifold of M. If Y_1 and Y_2 are smooth vector fields on M that are tangent to S, then $[Y_1, Y_2]$ is also tangent to S.*

Proof.

1. This is just the special case of Proposition 8.20 in which F is a diffeomorphism and $Y_i = F_*X_i$.
2. We know there exist smooth vector fields X_1 and X_2 on S such that X_i is ι-related to Y_i for $i = 1, 2$, where $\iota : S \to M$ is the inclusion. By Proposition 8.20, $[X_1, X_2]$ is ι-related to $[Y_1, Y_2]$, which is therefore tangent to S.

□