

8. Vector Fields

8.1 Smooth vector fields

Definition 8.1. Let M be a smooth manifold. A **vector field on M** is a continuous map $X : M \rightarrow TM$, usually written $p \mapsto X_p$, with the property that

$$\pi \circ X = \text{id}_M, \quad \pi : TM \rightarrow M \text{ the projection map} \quad (8.1)$$

or equivalently, $X_p \in T_pM$ for each $p \in M$.

One can visualize a vector field on M in the same way as to visualize vector fields in Euclidean space: as an arrow attached to each point of M , chosen to be tangent to M and to vary continuously from point to point.

Since we already saw that if M is a smooth manifold then the tangent bundle TM inherits a smooth structure from M in Theorem 4.8, we say that a vector field is a **smooth vector field** if it is smooth as map from M to TM .

Just as for functions, if X is a vector field on M , the **support of X** is defined to be the closure of the set $\{p \in M : X_p \neq 0\}$. A vector field is said to be **compactly supported** if its support is a compact set.

Suppose M is a smooth n -manifold. If $X : M \rightarrow TM$ is a vector field and $(U, (x^i))$ is any smooth coordinate chart for M , we can write the value of X at any point $p \in U$ in terms of the coordinate basis vectors:

$$X_p = X^i(p) \left. \frac{\partial}{\partial x^i} \right|_p. \quad (8.2)$$

This defines n functions $X^i : U \rightarrow \mathbb{R}$, called the **component functions of X** in the given chart. We can give the smoothness of vector fields in terms of the smoothness of the component functions.

Proposition 8.2. Let M be a smooth manifold, and let $X : M \rightarrow TM$ be a vector field. If $(U, (x^i))$ is any smooth coordinate chart on M , then the restriction of X to U is smooth if and only if its component functions with respect to this chart are smooth.

Proof. Let (x^i, v^i) be the natural coordinates on $\pi^{-1}(U) \subseteq TM$ associated with the chart $(U, (x^i))$. By definition of natural coordinates, the coordinate representation of $X : M \rightarrow TM$ on U is

$$\widehat{X}(x) = (x^1, \dots, x^n, X^1(x), \dots, X^n(x)),$$

where X^i is the i -th component function of X in x^i -coordinates. It follows immediately that smoothness of X in U is equivalent to smoothness of its component functions. \square

Let's see some examples of smooth vector fields.

Example 8.3. (Coordinate Vector Fields). If $(U, (x^i))$ is any smooth chart on M , the assignment

$$p \mapsto \left. \frac{\partial}{\partial x^i} \right|_p$$

determines a vector field on U , called the **i -th coordinate vector field** and denoted by $\partial/\partial x^i$. It is smooth because its component functions are constants.

Example 8.4. (The Euler Vector Field). The vector field V on \mathbb{R}^n whose value at $x \in \mathbb{R}^n$ is

$$V_x = x^1 \left. \frac{\partial}{\partial x^1} \right|_x + \dots + x^n \left. \frac{\partial}{\partial x^n} \right|_x$$

is smooth because its coordinate functions are linear. It vanishes at the origin, and points radially outward everywhere else. It is called the **Euler vector field**.

If $U \subseteq M$ is open, the fact that T_pU is naturally identified with T_pM for each $p \in U$ allows us to identify TU with the open subset $\pi^{-1}(U) \subseteq TM$. Therefore, a vector field on U can be thought of either as a map from U to TU or as

a map from U to TM , whichever is more convenient. If X is a vector field on M , its restriction $X|_U$ is a vector field on U , which is smooth if X is.

If M is a smooth manifold we would use

Notation: the notation $\mathfrak{X}(M)$ or $\Gamma(TM)$ to denote the set of all smooth vector fields on M . The set $\mathfrak{X}(M)$ is a \mathbb{R} -vector space under point wise addition and scalar multiplication:

$$(aX + bY)_p = aX_p + bY_p.$$

The zero element of this vector space is the zero vector field, whose value at each $p \in M$ is $0 \in T_pM$. In addition, smooth vector fields can be multiplied by smooth real-valued functions: if $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, we define $fX : M \rightarrow TM$ by

$$(fX)_p = f(p)X_p.$$

The next proposition shows that these operations indeed yield smooth vector fields.

Proposition 8.5. *Let M be a smooth manifold.*

- (1) *If X and Y are smooth vector fields on M and $f, g \in C^\infty(M)$, then $fX + gY$ is a smooth vector field.*
- (2) *$\mathfrak{X}(M)$ is a module over the ring $C^\infty(M)$.*

Proof. **Exercise.** □

8.2 Local and Global Frames

Coordinate vector fields in a smooth chart provide a convenient way of representing vector fields, because their values form a basis for the tangent space at each point. However, they are not the only choices.

Suppose M is a smooth n -manifold. An ordered k -tuple (X_1, \dots, X_k) of vector fields defined on some subset $A \subseteq M$ is said to be **linearly independent** if $(X_1|_p, \dots, X_k|_p)$ is a linearly independent k -tuple in T_pM for each $p \in A$, and is said to **span the tangent bundle** if the k -tuple $(X_1|_p, \dots, X_k|_p)$ spans T_pM at each $p \in A$.

Definition 8.6. A **local frame** for M is an ordered n -tuple of vector fields (E_1, \dots, E_n) defined on an open subset $U \subseteq M$ that is linearly independent and spans the tangent bundle. In other words, the vectors $(E_1|_p, \dots, E_n|_p)$ form a basis for T_pM at each $p \in U$.

It is called a **global frame** if $U = M$, and a **smooth frame** if each of the vector fields E_i is smooth.

Example 8.7. (Local and Global Frames).

- (a) The standard coordinate vector fields form a smooth global frame for \mathbb{R}^n .
- (b) If $(U, (x^i))$ is any smooth coordinate chart for a smooth manifold M , then the coordinate vector fields form a smooth local frame $(\partial/\partial x^i)$ on U , called a **coordinate frame**. Every point of M is in the domain of such a local frame.
- (c) The vector field $d/d\theta$ on S^1 where θ is the angle coordinate is a smooth global frame for the circle.
- (d) The n -tuple of vector fields $(\partial/\partial\theta^1, \dots, \partial/\partial\theta^n)$ on the n -torus $T^n = S^1 \times \dots \times S^1$ is a smooth global frame for T^n .

For subsets of \mathbb{R}^n , there is a special type of frame that is often more useful for geometric problems than arbitrary frames. A k -tuple of vector fields (E_1, \dots, E_k) defined on some subset $A \subseteq \mathbb{R}^n$ is said to be **orthonormal** if for each $p \in A$, the vectors $(E_1|_p, \dots, E_k|_p)$ are orthonormal with respect to the Euclidean dot product (where we identify $T_p\mathbb{R}^n$ with \mathbb{R}^n in the usual way). A (local or global) frame consisting of orthonormal vector fields is called an **orthonormal frame**.

8.3 Vector fields and derivations

Recall that we can view a tangent vector as a derivation. Similarly, vector fields define operators on the space of smooth real-valued functions $C^\infty(M)$ on a manifold M . If $X \in \mathfrak{X}(M)$ and f is a smooth real-valued function defined on an open subset $U \subseteq M$, we obtain a new function $Xf : U \rightarrow \mathbb{R}$, defined by

$$(Xf)(p) = X_p f,$$

where the tangent vector X_p is acting as a derivation on the function f . Since the action of a tangent vector on a function is determined by the values of the function in an arbitrarily small neighbourhood, it follows that Xf is locally determined. In particular, for any open subset $V \subseteq U$,

$$(Xf)|_V = X(f|_V). \tag{8.3}$$

This construction yields another useful smoothness criterion for vector fields.

Proposition 8.8. *Let M be a smooth manifold and let $X : M \rightarrow TM$ be a vector field. The following are equivalent:*

- (a) X is smooth.
- (b) For every $f \in C^\infty(M)$, the function Xf is smooth on M .
- (c) For every open subset $U \subseteq M$ and every $f \in C^\infty(U)$, the function Xf is smooth on U .

Proof. We will prove that (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a).

To prove (a) \Rightarrow (b), assume X is smooth, and let $f \in C^\infty(M)$. For any $p \in M$, we can choose smooth coordinates (x^i) on a neighbourhood U of p . Then for $x \in U$, we can write

$$Xf(x) = \left(X^i(x) \frac{\partial}{\partial x^i} \Big|_x \right) f = X^i(x) \frac{\partial f}{\partial x^i}(x).$$

Since the component functions X^i are smooth on U , it follows that Xf is smooth in U . Since the same is true in a neighbourhood of each point, Xf is smooth on M .

To prove (b) \Rightarrow (c), suppose $U \subseteq M$ is open and $f \in C^\infty(U)$. For any $p \in U$, let ψ be a smooth bump function that is equal to 1 in a neighbourhood of p and supported in U , and define $\tilde{f} = \psi f$, extended to be zero on $M \setminus \text{supp } \psi$. Then $X\tilde{f}$ is smooth by assumption, and is equal to Xf in a neighbourhood of p . This shows that Xf is smooth in a neighbourhood of each point of U .

Finally, to prove (c) \Rightarrow (a), suppose Xf is smooth whenever f is smooth on an open subset of M . If (x^i) are any smooth local coordinates on $U \subseteq M$, we can think of each coordinate x^i as a smooth function on U . Applying X to one of these functions, we obtain

$$Xx^i = X^j \frac{\partial}{\partial x^j}(x^i) = X^i.$$

Because Xx^i is smooth by assumption, it follows that the component functions of X are smooth, so X is smooth. \square

Thus, a smooth vector field $X \in \mathfrak{X}(M)$ defines a map from

$$C^\infty(M) \rightarrow C^\infty(M), \quad f \mapsto Xf.$$

This map is clearly linear over \mathbb{R} . Moreover, the Leibniz rule for tangent vectors as derivations implies the product rule for vector fields:

$$X(fg) = fXg + gXf, \tag{8.4}$$

as you can easily check by evaluating both sides at an arbitrary point $p \in M$.

A map $X : C^\infty(M) \rightarrow C^\infty(M)$ is called a **derivation** (as distinct from a *derivation at p*) if it is linear over \mathbb{R} and satisfies (8.4) for all $f, g \in C^\infty(M)$. Thus, any vector field is a derivation on $C^\infty(M)$.

The function Xf is sometimes also denoted by $\mathcal{L}_X f = Xf$ and is called the **Lie derivative of f in the direction of X** .

The next proposition shows that derivations of $C^\infty(M)$ can be identified with smooth vector fields.

Lemma 8.9. *Let M be a smooth manifold. A map $D : C^\infty(M) \rightarrow C^\infty(M)$ is a derivation if and only if it is of the form $Df = Xf = \mathcal{L}_X f$ for some smooth vector field $X \in \mathfrak{X}(M)$.*

Proof. We just showed above that every smooth vector field induces a derivation. Conversely, suppose $D : C^\infty(M) \rightarrow C^\infty(M)$ is a derivation. We need to come up with a vector field X such that $Df = Xf$ for all f . From the discussion above, it is clear that if there is such a vector field, its value at $p \in M$ must be the derivation at p whose action on any smooth real-valued function f is given by

$$X_p f = Df(p).$$

The linearity of D guarantees that this expression depends linearly on f , and the fact that D is a derivation yields the Leibniz rule for tangent vectors. Thus, the map $X_p : C^\infty(M) \rightarrow \mathbb{R}$ so defined is indeed a tangent vector, that is, a derivation of $C^\infty(M)$ at p . This defines X as a vector field. Because $Xf = Df$ is smooth whenever $f \in C^\infty(M)$, this vector field is smooth by Proposition 8.8. \square

Thus,

$$\text{smooth vector fields on } M \leftrightarrow \text{derivations of } C^\infty(M).$$

8.4 Vector fields, smooth maps, pushforwards and pullbacks

Recall that if $F : M \rightarrow N$ is a smooth map and X is a vector field on M , then for each point $p \in M$, we obtain a vector $dF_p(X_p) \in T_{F(p)}N$ by applying the differential of F to X_p . This begs the question: Can this process when applied globally, gives a vector field on N instead of just tangent vectors which depend on the point? The answer is **no**, this does not in general define a *vector field* on N . For example, if F is not surjective, there is no way to decide what vector to assign to a point $q \in N \setminus F(M)$. If F is not injective, then for some points of N there may be several different vectors obtained by applying dF to X at different points of M .

Suppose $F : M \rightarrow N$ is smooth and X is a vector field on M , and suppose there happens to be a vector field Y on N with the property that for each $p \in M$, $dF_p(X_p) = Y_{F(p)}$. In this case, we say the vector fields X and Y are **F-related**. The next proposition shows how F -related vector fields act on smooth functions.

Proposition 8.10. *Suppose $F : M \rightarrow N$ is a smooth map between manifolds, $X \in \mathfrak{X}(M)$, and $Y \in \mathfrak{X}(N)$. Then X and Y are F -related if and only if for every smooth real-valued function f defined on an open subset of N ,*

$$X(f \circ F) = (Yf) \circ F. \quad (8.5)$$

Proof. For any $p \in M$ and any smooth real-valued f defined in a neighbourhood of $F(p)$,

$$X(f \circ F)(p) = X_p(f \circ F) = dF_p(X_p)f,$$

while

$$(Yf) \circ F(p) = (Yf)(F(p)) = Y_{F(p)}f.$$

Thus, (8.5) is true for all f if and only if $dF_p(X_p) = Y_{F(p)}$ for all p , i.e., if and only if X and Y are F -related. \square

Example 8.11. Let $F : \mathbb{R} \rightarrow \mathbb{R}^2$ be the smooth map $F(t) = (\cos t, \sin t)$. Then $\frac{d}{dt} \in \mathfrak{X}(\mathbb{R})$ is F -related to the vector field $Y \in \mathfrak{X}(\mathbb{R}^2)$ defined by

$$Y = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}.$$