Lecture 1

Before diving deep into the study of differential geometry, let us try to discuss informally what a study of differential geometry comprises of. Suppose $\Sigma^2 \subset \mathbb{R}^3$ is a "smooth surface" and let $x, y \in \Sigma^2$ and we would like to measure "distance" between x and y. Naturally, we would like to measure the distance between two point on Σ *along* Σ. For the time being, suppose we have a path $\gamma : [a, b] \to \mathbb{R}^3$ with $\gamma([a.b]) \subset \Sigma$, $\gamma(a) = x$ and $\gamma(b) = y$. We can then define the "length of γ" by

$$l(\gamma) = \int_a^b |\dot{\gamma}(t)|dt = \int_a^b \sqrt{\langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle}dt \tag{0.1}$$

where $\dot{\gamma}(t) = \frac{d}{dt}\gamma(t)$, $\langle\cdot, \cdot\rangle$ is the Euclidean inner product and $|v| = \sqrt{\langle v, v\rangle}$. Having defined the length, we can define the distance between x and y as

$$d(x, y) := \inf_{P(x,y)} l(\gamma), \tag{0.2}$$

where $P(x, y) = \{\gamma \mid \gamma \text{ is a path between } x \text{ and } y\}$.

This gives rise to the following question:

Question 0.1. *Given a smooth surface Σ^2 and $x, y \in \Sigma$, does $\exists$ a smooth path from x to y that has the shortest possible length? If yes, then is such a path unique?*

We will see later in the course that the answer to both the questions is yes if x and y are close enough. Such a shortest path is called a **geodesic** (**Geodäte**) and is characterized by a second-order ordinary differential equation (gewöhnliche Differentialgelichung zweiter ordnung.) Such geodesics are the analogoues of "straight lines" on Σ even when no *straight path* might exist on Σ. For instance, you might already know that the shortest path between any two points on the surface of a Sphere (Sphäre), or in other words geodesics, are "great circles" (Großkreise).

One thing which you might have noticed is that the main concept which we needed to define the length of a path and in turn, the distance between two points is that of the Euclidean inner product $\langle, \rangle$. We needed to know the value of $\langle v, w\rangle$ for those $v, w \in \mathbb{R}^3$ which are tangent to Σ at any given point. Again, you might know from your previous courses that $\langle, \rangle$ allows us to compute **angles** (**Winkel**)θ between two vectors v, w by the formula

$$\langle v, w\rangle = |v||w| \cos\theta.$$

Thus, we can not only define length of any smooth path along Σ but also the angle between two smooth paths if they intersect. The procedure of being able to take inner product of two "tangent vectors" (Tangentialvektoren) makes Σ a 2**-dimensional Riemannian manifold** (**2-dimensionale Riemannsche Mannigfaltigkeit**) and the inner product will be called a **Riemannian metric** (**Riemannsche metrik**). As with any new mathematical object/structures, a natural question arises that suppose $\Sigma_1, \Sigma_2 \subset \mathbb{R}^3$ are two smooth surfaces and $f : \Sigma_1 \to \Sigma_2$ a smooth bijective map (bijektive Abbildung) such that $f^{-1} : \Sigma_2 \to \Sigma_2$ is also smooth. Such a function f is called a **diffeomorphism** (**Diffeomorphismus**) and Σ_1 and Σ_2 are called **diffeomorphic** (**diffeomorph**) to each other. In addition, if f preserves all distances and angles then it is called an **isometry** (**Isometrie**) and Σ_1 and Σ_2 are then **isometric** (**isometrisch**) to each other.

Question 0.2. *If Σ_1 and Σ_2 are two diffeomorphic surfaces, is there any way to know if they are also isometric?*

For example, consider the two different spheres in $\mathbb{R}^3$. In Figure 1, we have the standard embedding of sphere in $\mathbb{R}^3$ and in Figure 2 we have a slightly nonstandard embedding with a part of sphere elongated so as to look like a cylinder. Intutively, it look like the two spheres are diffeomorphic to each other. Again, intuitively, it feels that they should *not* be isometric as the length of paths on the parts of the spheres in the elongated portion might be different. But how do we make this idea rigorous?

One way to make this idea rigorous is to introduce the concept of parallel transport (Paralleltransport) of vectors along closed paths. In the figures, Figure 1 has a "rounder" sphere and the parallel transport of a vector v along the drawn closed path gives a different vector on return. On the contrary, the shaded part of the sphere in Figure 2 looks flat and the parallel transport of the vector v is the same vector in the end. This will be made rigorous in the coming lectures and we will show that this property is related to the **curvature** (**Krümmung**) of the sphere at a point. This will also allow us to prove that the spheres in both the figures are indeed not isometric!

We will spend a lot of time in defining and studying the properties of curvature where we will prove that it is a mathematical object called a **tensor field** (**Tensorfeld**). We will also see the notion of *local flatness*, i.e., parts of surfaces which locally look like parts of $\mathbb{R}^2$. For instance, the cylinder in Figure 3 shows a cylinder which is locally flat. In fact, this begs the question that can the *whole* sphere in Figure 2 be elongated so that the sphere is not curved at all?

This will be answered after we will prove the following amazing result.

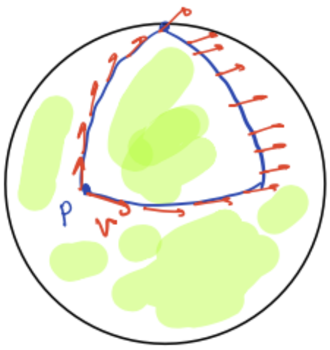


Figure 1: Standard embedding of $S^2 \subset \mathbb{R}^3$ with parallel transport of a vector along a closed path leading to a different vector on return.

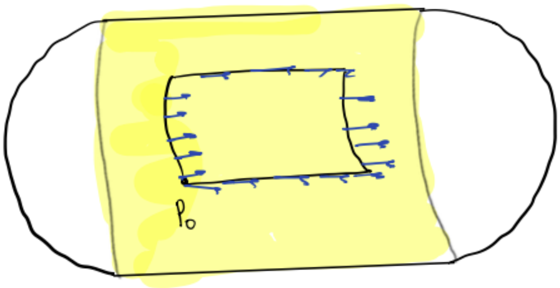


Figure 2: Another embedding of $S^2 \subset \mathbb{R}^3$ but now the yeellow shaded region is locally flat. Parallel transport of a vector returns to the same vector along a closed path.

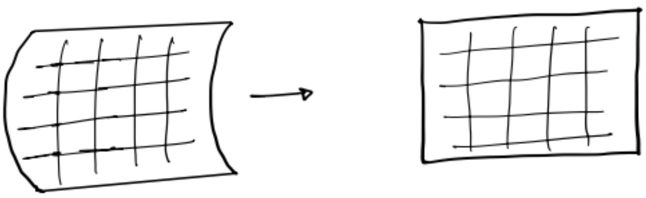


Figure 3:

Theorem 0.3 (Gauss–Bonnet theorem)**.** *If M is a compact, 2-dimensional Riemannian manifold with boundary ∂M and if K is the Gauss curvature of M and k is the geodesic curvature of ∂M, then*

$$\int_M K dA + \int_{\partial M} k ds = 2\pi\chi(M), \tag{0.3}$$

where $\chi(M)$ is the Euler characteristic of M.

We will use the Gauss–Bonnet theorem to prove that there cannot exist a Riemannian metric on S^2 which is everywhere locally flat.

We now start studying the topics formally.

1. Manifolds

1.1 Topological manifolds

We first recall some basic notions from topology.

Definition 1.1. Let M be a set. A subset $\mathcal{O} \subset \mathcal{P}(M)$ is called a **topology** on M if

1. $\emptyset, M \in \mathcal{O}$.
2. Arbitrary unions of elements of $\mathcal{O}$ is in $\mathcal{O}$, ie., if $\{U_i\}_{i\in I} \in \mathcal{O}$ then so is $\underset{i\in I}{\cup} U_i$.
3. Finite intersections of elements in $\mathcal{O}$ are in $\mathcal{O}$, i.e., if $U_1, U_2 \in \mathcal{O}$ then $U_1 \cap U_2 \in \mathcal{O}$. We call the pair $(M, \mathcal{O})$ a **topological space.** (**topologischen Raum**)

Elements of $\mathcal{O}$ are called **open sets** (**offene Mengen**). A subset $\subset M$ is called **closed** (abgeschlossen) is $M \setminus C \in \mathcal{O}$. Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be two topological spaces. A map $f : M \to N$ is called **continuous** (**Stetig**), if

$$f^{-1}(V) \in \mathcal{O}_M \;\; \forall \;\; V \in \mathcal{O}_N.$$

This means, that preimages of open sets in N are open in M. A bijective continuous map $f : M \to N$ with f^{-1} also being continuous is called a **homeomorphism** (**Homöomorphismus**), in which case M and N are said to be **homoemorphic** (**homömorph**).

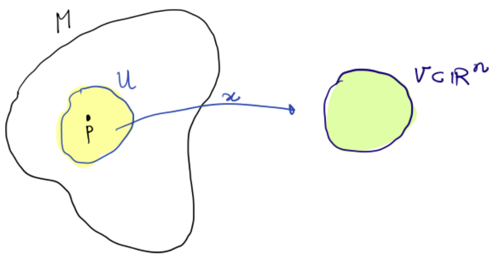
We can now define what a topological manifold is.

Definition 1.2. Let $(M, \mathcal{O})$ be a topological space. Then M is called an **n-dimensional topological manifold** (**n-dimensionale topologische Mannigfaltigkeit**) if

1. M is **Hausdorff** (**hausdorffsch**), ie., every pair of distinct points can be separated by open sets: given $p, q \in M$, there exists open sets $U \ni p$ and $V \ni q$ such that $U \cap V = \emptyset$.
2. The topology of M has a **countable basis** (**abzählbare Basis**), which means that there exists a countable subset $\mathcal{B} \subset \mathcal{O}$ such that for every $U \in \mathcal{O}$, there exists $B_i \in \mathcal{B}$, $i \in I$ with
$$U = \bigcup_{i\in I} B_i.$$
3. (most important) M is **locally homeomorphic** (**lokal homöonorph**) to $\mathbb{R}^n$ (same n as in the definition), that is, for all $p \in M$, there exists an open set $U \ni p$ and an open set $V \subset \mathbb{R}^n$ and a homeomorphsim between $x : U \to V$ between them.

   This basically means that any manifold locally looks an Euclidean space. A pictorial representation is shown in the figure below.

Definition 1.3. The homeomorphisms $x : U \to V$ in Definition 1.2 are called **charts** (**Karten**) or **local coordinate systems** of M (**lokale Koordinatensysteme**).



Example 1.4. Any Euclidean space $\mathbb{R}^n$ is a n-dimensional manifold with $U = M$, $V = \mathbb{R}^n$ and $x = id$, the identity map.

Example 1.5. Any open set $O \subset \mathbb{R}^n$ is an n-dimensional manifold with $U = O$, $V = \mathbb{R}^n$ and $x = id$.

Example 1.6. The n-dimensional sphere

$$S^n = \{y \in \mathbb{R}^n \mid ||y|| = 1\}$$

is a n-dimensional manifold. The existence of charts can be proved using the stereographic projection. If $s = \{-1, 0, \ldots, 0\} \in \mathbb{R}^{n+1}$ denotes the south pole of the sphere then we define $U_1 = S^n \setminus \{s\}$ and let $V_1 = \mathbb{R}^n$. Define the map x by

$$x(y^0, \underbrace{y^1, \ldots y^n}_{\hat{y}}) = \frac{2}{1 + y^0} \cdot \hat{y}.$$

Clearly the map x is continuous and bijective. The inverse map x^{-1} is given by

$$x^{-1}(z) = \frac{1}{4 + ||z||^2}(4 - ||z||^2, 4z)$$

and hence is also continuous. Thus, x is a chart of M.

Similarly, when we remove the north pole $n = \{1, 0, \ldots 0\} \in bR^{n+1}$ and let $U_2 = S^n \setminus \{n\}$ and again choose $V_2 = \mathbb{R}^{n+1}$ then $x' : U_2 \to V_2$ given by

$$x'(y^0, \underbrace{y^1, \ldots, y^n}_{\hat{y}}) = \frac{2}{1 - y^0} \cdot \hat{y},$$

is another chart, thus proving that S^n is an n-dimensional manifold.

Exercise 1.7. Consider the double cone (Doppel-kegel) $M = \{(y^1, y^2, y^3) \in \mathbb{R}^3 \mid (y^1)^2 = (y^2)^2 + (y^3)^2\}$. Prove that M *is not* a 2-dimensional manifold by showing the failure of existence of charts.

Example 1.8. Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^k$ be a continuous function. The **graph of** f (**Grafik von** f) is the set

$$\Gamma(f) = \{(y, f(y)) \mid y \in U\}.$$

Let $\pi_1 : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ be the projection onto the first factor and let $x : \Gamma(f) \to U$ be $x = \pi_{1_{|\Gamma(f)}}$ which is a continuous bijection with a continuous inverse. Thus, $\Gamma(f)$ is an n-dimensional topological manifold and is in fact, homeomorphic to U itself. The chart $(\Gamma(f), x)$ are called graph coordinates.

If you assume the fact that $\mathbb{R}^n$ and $\mathbb{R}^m$ can be homeomorphic if and only if $n = m$ then we have the following

Lemma 1.9. *(Topological Invariance of Dimension). The dimension of a topological manifold is a topological invariant of the manifold.*

We will see more examples of manifolds. Suppose we are given a chart or local coordinates system $x : U \to V$. Then at every point $p \in U$, we can look at $(x^1(p), \dots, x^n(p)) \in V \subset \mathbb{R}^n$, which we will called **coordinates of** p (**Koordinaten von** p).

Before discussing more examples and properties of manifolds, let's see if we can gather more information about them in lower dimensions.

0-dimension. If a manifold M is 0-dimensional then every point $p \in M$ has an open neighbourhood $U \ni p$ which is homemorphic to $\mathbb{R}^0 = \{0\}$. Thus, $\{p\}$ is an open subset of M for any p and hence M has the discrete topology. Moreover, the second requirement in Definition 1.2 forces M to be a countable space. Thus, we have proved the following

Proposition 1.10. *The only topological manifold of dimension* 0 *are countable spaces with discrete topology.*

We will prove the following classification about 1-dimensional manifolds in future lectures.

Proposition 1.11. *Any connected* 1*-dimensional manifold is homeomorphic to either* S^1 *or* $\mathbb{R}$.

Below giving the next example, we would like to understand better the realtion between the topology on M and the set of charts on M. Intuitively, it looks like we should be able to tell everything about the topology of M if we are given a collection of charts of M which cover it. This is the content of the next lemma.

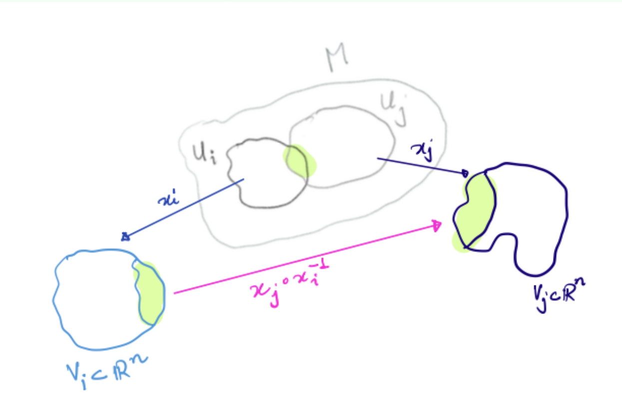
Lemma 1.12. *Every manifold has a unique topology with open sets being the domains of charts of the manifold.*

We prove this lemma in a series of propositions.

Proposition 1.13. *Let* M *be a set and* I *be an indexing set. Suppose for all* $i \in I$, *we have* $U_i \subset M$, $V_i \subset \mathbb{R}^n$ *and* $x_i : U_i \to V_i$ *are bijections. Suppose the following holds:*

1. $\bigcup_{i \in I} U_i = M$,
2. *for all* $i \in I$, $x_i(U_i \cap U_j) \subset \mathbb{R}^n$ *is open, and*
3. $x_j \circ x_i^{-1} : x_i(U_i \cap U_j) \to x_j(U_i \cap U_j)$ *is a continuous map on* $\mathbb{R}^n$, *for all* $i, j \in I$ *(see Figure 1.13).*

Then M *carries a unique topology where the sets* U_i *are open sets and all the maps* x_i *are homeomorphisms.*



If in addition, $\exists\, I'$ *which is countable such that* $\bigcup_{i \in I'} U_i = M$ *then the topology on* M *has a countable basis and if for any* $p, q \in M$ $\exists\, i \in I$ *with* $p, q \in U_i$ *then* M *is also Hausdorff.*

Proof of Proposition 1.13. We first prove that if such a topology exists on M then it must be unique. We prove this by proving that if $\mathcal{O}$ is such a topology then a set $X \in \mathcal{O} \iff x_i(X \cap U_i)$ is open in $\mathbb{R}^n$ for all $i \in I$. Suppose $\mathcal{O}$

is a topology on M such that U_i are all open sets and x_i are all homeomorphisms. Let $X \in \mathcal{O}$. Then $X \cap U_i \in \mathcal{O}$ and hence $x_i(X \cap U_i)$ is open in $\mathbb{R}^n$ for all i. Conversely, suppose $X \subset M$ such that $x_i(U_i \cap X) \subset \mathbb{R}^n$ is open for all i. This means that $x_i \circ x_i^{-1}(U_i \cap X)$ is open in $\mathbb{R}^n$ and hence X is open in M. Thus, if a toplogy $\mathcal{O}$ given by the conditions of the proposition exists then it must be unique.

To prove the existence, we show that the set

$$\mathcal{O} = \{X \subset M \mid x_i(X \cap U_i) \subset \mathbb{R}^n \text{ open } \forall\, i \in I\}$$

describes a topology on M. Clearly, $\emptyset, M \in \mathcal{O}$ as $x_i(M \cap U_i) = x_i(U_i)$ is open in $\mathbb{R}^n$ by condition 2. If $X_j \in \mathcal{O},\ j \in J$ then, for all $i \in I$, we have

$$x_i\left(\left(\bigcup_{j\in J} X_j\right) \cap U_i\right) = x_i\left(\bigcup_{j\in J}(X_j \cap U_i)\right) = \bigcup_{j\in J} \underbrace{x_i(X_j \cap U_i)}_{\text{open in } \mathbb{R}^n}$$

and hence arbitrary unions of sets in $\mathcal{O}$ are in $\mathcal{O}$. You should prove the finite intersection part yourself. Thus, $\mathcal{O}$ described as above is indeed a topology on M.

Now we want to show that in this topology, U_i are open sets. But this is clear by condition 2. We now prove that x_i are homeomorphisms for all $i \in I$. Let $x_i : U_i \to V_i \subset \mathbb{R}^n$ be a map for some $i \in I$. We only know part 3. Let $Y \subset V_i$ be open. Then, for any $j \in I$, we have

$$\begin{aligned} x_j(x_i^{-1}(Y) \cap U_j) &= x_j(x_i^{-1}(Y \cap x_i(U_i \cap U_j))) \\ &= \underbrace{(x_j \circ x_i^{-1})}_{\text{continuous}} \underbrace{Y \cap \underbrace{x_i(U_i \cap U_j)}_{\text{open by part 2.}}}_{\text{open}}, \end{aligned}$$

is open in $\mathbb{R}^n$ and hence by the definition of $\mathcal{O}$, $x_i^{-1}(Y) \in \mathcal{O}$. Similarly, you can prove that x_i^{-1} is also continuous, which proves the proposition.

Now suppose the indexing set $I' \subset I$ is countable which covers M. Since the topology is unique, I and I' give the same topology on M. Now the topology of each $V_i \subset \mathbb{R}^n$ has a countable basis B_i. Then $x_i^{-1}(B_i)$is a countable basis of the topology of U_i and thus, $\bigcup_{i\in I'} x_i^{-1}B_i$ is a countable basis of M.

Finally, Let $p, q \in M$ with $p \neq q$ and let $i \in I$ such that $p, q \in U_i$. Since $\mathbb{R}^n$ is Hausdorff, we can choose $V_1, V_2 \subset V_i$ open with $x_i(p) \in V_1$, $x_i(q) \in V_2$ and $V_1 \cap V_2 = \emptyset$. Then $x_i^{-1}(V_1)$ and $x_i^{-1}(V_2)$ separate p and q. □

Example 1.14 (Real Projective Space). We define the real projective space (reeller projektiver Raum) by

$$\mathbb{RP}^n = \{L \subset \mathbb{R}^{n+1} \mid L \text{ is one-dimensional subspace}\}, \tag{1.1}$$

that is, $\mathbb{RP}^n$ is the set of lines in $\mathbb{R}^{n+1}$. We use the quotient topology (Quotiententopologie) on $\mathbb{RP}^n$ by using the quotient map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ which sends each point p in $\mathbb{R}^{n+1} \setminus 0$ to the subspace spanned by p, in other words, we use the equivalence relation $p \sim \lambda p, \lambda \neq 0 \in \mathbb{R}$.

We exhibit charts for $\mathbb{RP}^n$. For each $i = 1, \dots, n+1$, let

$$\overline{U}_i \subset \mathbb{R}^{n+1} \setminus \{0\} \text{ where } p^i \neq 0,$$

and let

$$U_i = \pi(\overline{U}_i).$$

By the properties of quotient maps, $U_i \subset \mathbb{RP}^n$ is open and we define $\phi_i : U_i \to \mathbb{R}^n$ by

$$\phi_i\left([p^1, \dots, p^{n+1}]\right) = \left(\frac{p^1}{p^i}, \dots, \frac{p^{i-1}}{p^i}, \frac{p^{i+1}}{p^i}, \dots, \frac{p^{n+1}}{p^i}\right).$$

This map is well-defined. Since $\phi_i \circ \pi$ is continuous, so is ϕ_i and in fact, it is a homeomorphism with the the continuous inverse given by

$$\phi_i^{-1}(u^1, \dots, u^n) = [u^1, \dots, u^{i-1}, 1, u^i, \dots u^n].$$

Geometrically, $\phi([p]) = u$ means $(u, 1)$ is the point in $\mathbb{R}^{n+1}$ where the line $[p]$ intersects the hyperplane where $p^i = 1$. Now the sets $U_1, \dots, U_{n+1}$ cover $\mathbb{RP}^n$, we see that (U_i, ϕ_i) form a chart of $\mathbb{RP}^n$, making it into a n-dimensional topological manifold.