Differential Geometry - I
Universität Hamburg, Summer Semester 2026

Instructor/Dozent :	Prof. Dr. Shubham Dwivedi Geomatikum Room 320 shubham.dwivedi@uni-hamburg.de
Website:	https://s-dwivedi.github.io/DG1S26.html
Moodle:	https://lernen.min.uni-hamburg.de/course/view.php?id=6578 enrolment key: riemann Moodle will be mostly used to upload your solutions to the problem sets and for grading.
Lectures/Vorlesungen:	Mondays 10:15 - 11:45 in MIN-F Hörsaal 7 Thursdays 10:15 - 11:45 in MIN-F Hörsaal 7.
Problem Sessions/Übungen:	Thursdays 14:15 - 15:45 in MIN-F SemRm 3.4.
Language:	German/English (tbd).

Course Description

Differential geometry, as the name suggests, is the combination of differential calculus and geometry. It begins with the study of smooth 1-dimensional curves and 2-dimensional surfaces in Euclidean space, but these are only special cases of much more general objects, called smooth n-dimensional manifolds. Manifolds arise naturally in many branches of mathematics as well as in physics,e.g. as the curved spacetime in Einstein's theory of gravitation (General Relativity). This course will study manifolds from a fairly general perspective, without limiting the discussion to curves and surfaces, though for the purposes of visualization, most of the examples we consider will be 2-dimensional or 3-dimensional. The course will start with trying to understand basic notions such as smoothness of maps between manifolds, the derivatives of such maps, tangent vectors, vector fields and the flows that they generate. Tensors are then introduced as a linear-algebraic means of encoding local geometric information, and as a special case, we consider differential forms, which define notions of volume on manifolds and can thus be integrated. The first portion of the course culminates with the general version of Stokes' theorem for integrals of differential forms on manifolds: this is the natural n-dimensional generalization of the fundamental theorem of calculus, and also implies the standard vector calculus theorems of Gauss, Green and Stokes.

The second half of the course is based on the general notion of a vector bundle, of which several examples (e.g. the set of all tangent or cotangent vectors on a manifold) will already be familiar from the first half.We will talk about the notions of parallel transport, connections and covariant differentiation. This raises a natural question as to when covariant derivatives in different directions can be assumed to commute, and the answer requires the introduction of curvature, a tensor whose vanishing characterizes the existence of "covariantly constant" vector fields on manifolds. In order to prove this, we introduce smooth distributions on manifolds and prove the Frobenius integrability theorem. The most convincing initial applications of these ideas are in the study of Riemannian manifolds: these are manifolds equipped with extra structure so that lengths of paths and angles between them can be defined. We consider geodesics, which define shortest paths between nearby points on Riemannian manifolds, and discuss the geometric meaning of the Riemann curvature tensor in n dimensions, as well as its simpler variant for surfaces, the so-called "Gaussian" curvature. We can then prove one of the most beautiful and fundamental results about 2-dimensional Riemannian manifolds: the Gauss-Bonnet theorem, which relates the sum of the angles in a geodesic triangle to the amount of curvature it encloses, or for the case of compact surfaces without boundary, computes the total curvature in terms of a purely topological invariant, the Euler characteristic.

Topics to be covered (these might not be in the same order as mentioned.)

(1) Basic notions from topology, definition of manifolds.
(2) Differentiable manifolds, implicit function theorem, examples, tangent vectors.
(3) Tangent maps, vector fields, Lie bracket and commuting flows.
(4) Orientability, tensors, index notation, differential forms.
(5) Lie derivative, Cartan's magic formula.
(6) Partitions of unity, existence of volume forms and Riemannian metrics.
(7) Integration, Stokes' theorem, low-dimensional examples.
(8) Linear connections, covariant derivatives, compatibility.
(9) Connections on tangent bundles, torsion and symmetry, geodesics, Riemann normal coordinates.
(10) Levi-Civita connection, geodesics, Riemannian manifolds.
(11) Integrability and the Frobenius theorem, curvature.
(12) Locally flat manifolds, hypersurfaces, second fundamental form and Gaussian curvature.
(13) Euler characteristic and the Gauss-Bonnet theorem for surfaces.
(14) Sectional curvature, second variation formula, length-minimizing geodesics.
(15) Vector bundles and sections, bundle metrics, orientation.

Literature

There are excellent introductions to the subject of the Differential geometry and the materials presented in the class will be followed from the references mentioned below. I will upload lecture notes on the course webpage.

- John M. Lee, Introduction to Smooth Manifolds, second edition, Springer GTM 2012.
- Michael Spivak, A Comprehensive Introduction to Differential Geometry, Volume I, 3rd edition with corrections, Publish or Perish 2005.
- Ilka Agricola and Thomas Friedrich, Globale Analysis: Differentialformen in Analysis, Geometrie und Physik, Vieweg 2001 (or the English translation, AMS 2002).
- Helga Baum, Differentialgeometrie, lecture notes, in German, available at https://www.mathematik.hu-berlin.de/ baum/Skript/DG-05-06.15.pdf .
- Chris Wendl, Differential Geometry 1 and 2, lecture notes available at https://www.mathematik.hu-berlin.de/ wendl/Sommer2022/Diffgeo2/lecturenotes.pdf .

Grading and Problem Sets

Grades in this course will be determined by a **3-hour written exam** soon after the end of the semester. The exam problems will be conceived so as to be solvable within 2 hours, so that time pressure should not be the decisive factor.

Problem sets will be posted on the course website every Thursday, and solutions discussed in the problem session on the following Thursday. **The problem sets will be graded**, and only the ∗-marked problems in the problem sets will be graded. However, it is strongly recommended that you at least think through every problem before the problem session each week, since this is the single best way to ensure that you are keeping up with the material in the course.

A minimum of 50 % of the total marks in the graded problems will be required to be allowed to take the exam. As such, it will be important that you solve and hand in your solutions.

Use of AI tools - I do not have any problem with using AI tools and platforms to understand problems and their solution *as long as you understand it yourself.* The final solutions must be written by you.